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Geometry on the Cuspidal Cubic Cone.

BY FREDERICK C. FERRY.

The geometry on the general cubic surface has attracted the attention of many mathematicians, and papers on the subject have been published by *Clebsch, †Diekmann, ‡Sturm, §Rohn, and others. The geometry on the cubic cone seems to have received by itself little consideration, which may be due to the fact that no great difficulty is found in applying to these particular surfaces the results obtained for the general cubic.

The following paper has for its object to investigate by the methods of analytic geometry the question of the order of the surface of lowest order which can be passed through any given proper curve on the cuspidal cubic cone, and also the character of the residual intersection in each case.

A system of coördinates, λ, μ, ν , is selected for the representation by homogeneous equations of curves on the cone in question, the degree of any such equation in λ and μ being denoted by p and its degree in ν by q . Any curve, whose equation is so represented, is found to pass $p - q$ times through the vertex of the cone, and to meet every element of the cone in q points apart from that vertex. The order of the curve as a curve in space is proved to be $p + 2q$; and this order is congruent, modulus 3, to the order of multiplicity of the vertex of the cone as a point of the curve.

The investigation of the order of the surface S of lowest order (save in general the given cone itself), which can contain the curve considered, is then taken up, and it is demonstrated that that order is as low as p in every case, while the

* Kronecker J., LXV, 359-380

† Math. Annalen, IV, 442-475.

‡ Math. Annalen, XXI, 457-515,

§ Leipz. Ber., XLVI, 84-119.

residual intersection is made up entirely of the cuspidal edge of the given cone occurring $2(p - q)$ times; but, if $p = q$, the curve is one of total intersection. If, however, $p > q + 2$, the order of this surface S can be made lower than p by means of the substitution of x^2z for y^3 in the equation of S , which renders that equation factorable by a power of x ; thus it is proved that this order of S can be so far reduced in every case that the residual intersection becomes of order not greater than four; and the residual intersection is still made up entirely of the cuspidal edge of the given cone occurring the requisite number of times.

It is then pointed out that a curve given by an equation of entirely general form in the coördinates chosen passes $2q$ times through the infinite point of the cuspidal edge of the given cone, and meets that edge nowhere else except at the vertex of the cone; accordingly, the special form of equation necessary for the representation of curves which meet that cuspidal edge elsewhere than at the two points mentioned, and the number of conditions imposed in each case are determined. Then the order of the surface S of lowest order containing the curves of this kind is ascertained; and it is proved that three of the four classes of curves here occurring are curves of total intersection, while in the case of each curve of the remaining class the residual intersection is made up entirely of the cuspidal edge of the given cone found twice only.

A table of results for the curves of the lower orders is given at the end of the paper.

The methods used throughout the paper are analogous to those employed by the same writer in papers on the geometry on the cubic scrolls.*

I.—*The Cuspidal Cubic Cone.*

The equation of the cuspidal cubic cone in homogeneous coördinates is

$$y^3 - x^2z = 0.$$

Let this cone, or the left member of its equation, as the case may demand, be denoted by Σ . The equation $x = 0$ gives the cuspidal tangent plane, which has contact with the cone all along the cuspidal edge and accordingly contains that

* "Geometry on the Cubic Scroll of the First Kind," *Archiv for Mathematik og Videnskab*, B. XXI, Nr. 3, p. 3-57. "Geometry on the Cubic Scroll of the Second Kind," *Amer. Journ. Math.*, XXIII, p. 179-235.

edge three times; $y = 0$ gives the plane through both the cuspidal and the inflectional edges, containing the former edge twice and the latter edge once; $z = 0$ gives the inflectional tangent plane, which has contact with Σ all along the inflectional edge and hence contains that edge three times; and $s = 0$ gives the infinite plane, whose intersection with Σ is a cuspidal cubic curve, the "infinite cubic" in the geometry under consideration. The cuspidal edge of Σ is given by the equations $x = 0$, $y = 0$, and the inflectional edge by $y = 0$, $z = 0$; while $z = 0$, $s = 0$ are the equations of the inflectional tangent to the infinite cubic, $s = 0$, $x = 0$ give the cuspidal tangent to the infinite cubic, $y = 0$, $s = 0$ give a line through the cusp and the point of inflection of the infinite cubic, hence an infinite line, and $x = 0$, $z = 0$ give a line through the vertex of Σ but containing no other point of the cone. The vertex of Σ is given by the equations $x = 0$, $y = 0$, $z = 0$, the cusp of the infinite cubic by $x = 0$, $y = 0$, $s = 0$, and the point of inflection of the infinite cubic by $y = 0$, $z = 0$, $s = 0$; the point whose equations are $x = 0$, $z = 0$, $s = 0$ does not lie on Σ .

II.—*Coördinates on Σ .*

Taking for coördinates on Σ the variables λ , μ and ν , so chosen that $x/y = \lambda/\mu$ and $x/s = \lambda/\nu$, the equation $\Sigma = 0$ gives the following relation between the coördinates of the two systems, viz.

$$x : y : z : s = \lambda^3 : \lambda^2\mu : \mu^3 : \lambda^2\nu.$$

Then $\lambda = 0$ gives the cuspidal edge of Σ , $\mu = 0$ gives the inflectional edge of Σ , and $\nu = 0$ gives the intersection of Σ with the infinite plane, i. e., the infinite cubic. The vertex of Σ is given by the equations $\lambda = 0$, $\mu = 0$, the cusp of the infinite cubic by $\lambda = 0$, $\nu = 0$, and the point of inflection of the infinite cubic by $\mu = 0$, $\nu = 0$. Any point of the cuspidal edge of Σ is given by an equation of the form $\mu = k\nu$, where k is a constant; thus, if k have the value zero, the equation gives the vertex of Σ , and, if k have the value infinity, the equation gives the cusp of the infinite cubic. Similarly, any point of the inflectional edge is given by an equation of the form $\lambda = k\nu$; this equation gives the vertex of Σ if k has the value zero, and the point of inflection of the infinite cubic if k has the value infinity. And, in like manner, any point of the infinite cubic is determined by an equation of the form $\lambda = k\mu$; the point thus determined will be the cusp of the infinite cubic if k has the value zero, and the point of inflection of that cubic if k becomes infinite.

III.—*Curves on Σ .*

It is evident that any proper curve on Σ may be represented by an irreducible homogeneous equation in the coördinates λ , μ and ν . Let such an equation be designated by $\phi = 0$, and let the degrees of that equation in all three variables and in the variable ν respectively be denoted by p and q , which demands that $p \geq q$. The equation in question may then be arranged according to powers of ν and represented thus:

$$\phi \equiv \phi_p + \nu \cdot \phi_{p-1} + \nu^2 \cdot \phi_{p-2} + \dots + \nu^q \cdot \phi_{p-q} = 0,$$

and the curve given by this equation may be called *the curve (p, q)* .

Since the equation $\phi = 0$ is of degree $p - q$ in the variables λ , μ , it is evident that the curve (p, q) in question has $p - q$ points coincident at the point on Σ given by the equations $\lambda = 0$, $\mu = 0$, i. e., *the curve (p, q) passes $p - q$ times through the vertex of Σ* . These $p - q$ points at the vertex of Σ will be regarded as lying on the $p - q$ edges of Σ given by the equation $\phi_{p-q} = 0$, and not, in general, on the cuspidal edge of Σ ; if, however, ϕ_{p-q} has λ^{α_1} as a factor, then will α_1 of the $p - q$ points in question be regarded as lying on the cuspidal edge.

The cuspidal edge of Σ is given by the equation $\lambda = 0$; if the value zero be inserted in place of λ in the equation $\phi = 0$, there results in general an equation in μ , ν of degree p of the form $\mu^{p-q} \cdot \psi_q = 0$, where ψ_q is homogeneous in μ , ν ; the factor μ^{p-q} corresponds to the $p - q$ points of the curve at the vertex of Σ , while $\psi_q = 0$ gives the q values of μ/ν for the q remaining points where the curve meets the cuspidal edge of Σ ; hence, *the curve (p, q) meets the cuspidal edge of Σ in q points in addition to the $p - q$ points of the curve at the vertex of Σ* .

Similarly, the inflectional edge of Σ is given by the equation $\mu = 0$, and, on inserting that value for μ in the equation $\phi = 0$, there results in general an equation of the form $\lambda^{p-q} \cdot \psi'_q = 0$, where ψ'_q is homogeneous in λ , ν ; the factor λ^{p-q} corresponds to the $p - q$ points of the curve at the vertex of Σ , while $\psi'_q = 0$ gives the q values of λ/ν for the q remaining points where the curve (p, q) meets the inflectional edge of Σ ; hence, *the curve (p, q) meets the inflectional edge of Σ in q points in addition to the $p - q$ points of the curve at the vertex of Σ* . In the general case μ cannot be a factor of ψ_q and λ cannot be a factor of ψ'_q ; the special cases where such factors occur will be considered later.

Any plane through the cuspidal and inflectional edges of Σ will pass through the vertex of Σ and will contain the cuspidal edge twice and the inflectional edge once. Consequently, such a plane will contain $p - q$ points of the curve (p, q) at the vertex, $2q$ points of that curve along the cuspidal edge, and q points along the inflectional edge; thus this plane will contain $p - q + 2q + q = p + 2q$ points of the curve in question. Hence, in the general case, *the order of the curve (p, q) as a curve in space is $p + 2q$* . Let m denote this order of the curve (p, q) ; then, in general, $m = p + 2q$.

An edge or element of Σ is a straight line, hence of order unity as a curve in space; consequently, an edge of Σ is a curve $(1, 0)$, and as such is given by an equation of the form $a\lambda + b\mu = 0$. This equation gives the cuspidal edge if b/a has the value zero, and the inflectional edge if b/a is infinite. Since this equation of the edge does not involve the variable ν , it follows that an edge, in general, has $q = 0$ points, i. e., no points in common with the cuspidal edge or the inflectional edge, apart from the vertex of Σ . Thus, *a homogeneous equation $\phi_p(\lambda, \mu) = 0$ of degree p in the coördinates λ, μ represents p edges of Σ* . These p edges may be wholly or in part distinct or coincident, and all together may be regarded as making up a $(p, 0)$, as their equation $\phi_p = 0$ shows. Evidently, the only curve having $q = 0$ is that consisting of an edge or group of edges of Σ .

To determine the points of intersection of any edge, whose equation is $a\lambda + b\mu = 0$, with the curve (p, q) given by $\phi = 0$, it is necessary to give λ the value $\lambda = -b/a \cdot \mu$ in the equation $\phi = 0$; this equation then takes the form $\mu^{p-q} \cdot \psi_q'' = 0$, where ψ_q'' is a homogeneous polynomial in μ, ν of degree q ; the factor μ^{p-q} corresponds to the $p - q$ points of the curve at the vertex of Σ , and $\psi_q'' = 0$ gives q values of μ/ν corresponding to the q points where the curve meets the edge in question. Hence, *the curve (p, q) has q points on each edge of Σ , in addition to the $p - q$ points of the curve lying at the vertex of Σ* . Particular edges of Σ , as already implied, may have one or more of their q points of the curve lying at the vertex of Σ , but none of the $p - q$ points at the vertex of Σ is to be regarded as lying on more than a single one of the edges at that point.

Since $p + 2q$ is congruent to $p - q$, modulus 3, it follows that the order of the curve (p, q) as a space curve is congruent, modulus 3, to the number of times the curve in question passes through the vertex of Σ ; if π denote the order of multiplicity of the vertex of Σ as a point of the curve (p, q) , then m and π are connected by the relation $m \equiv \pi \pmod{3}$.

Two curves (p, q) and (p', q') , where $p = p'$ and $q = q'$, are said to be of the same *species*. The theorem stated above shows that the only possible curves, when $\pi \equiv 0 \pmod{3}$, are of the orders 3, 6, 9, 12, 15, etc., and of the species (1, 1), (2, 2), (3, 3), (3, 0), (4, 1), (4, 4), etc.; if $\pi \equiv 1 \pmod{3}$, the only possible curves are of the orders 1, 4, 7, 10, 13, 16, etc., and of the species (1, 0), (2, 1), (3, 2), (4, 0), (4, 3), (5, 1), etc., and, if $\pi \equiv 2 \pmod{3}$, the only possible curves are of the orders 2, 5, 8, 11, 14, 17, etc., and of the species (2, 0), (3, 1), (4, 2), (5, 0), (5, 3), (6, 1), etc.

It will be supposed, unless otherwise stated, that the equation $\phi = 0$ is irreducible and entirely general in form; consequently, that equation will not represent henceforth any curve $(p, 0)$ where p has a value greater than unity.

IV.—*The Curve (p, q) as the Intersection, Total or Partial, of a Surface S with Σ .*

To find the equation of a surface S cutting the curve (p, q) from Σ , it is necessary to substitute x, y, z and s in the equation of the curve (p, q) , according to the laws connecting the space-coördinates x, y, z, s with the Σ -coördinates λ, μ, ν ; i. e., $\lambda^3, \lambda^2\mu, \mu^3$ and $\lambda^2\nu$ are to be replaced respectively by x, y, z and s in accordance with the proportion

$$x : y : z : s = \lambda^3 : \lambda^2\mu : \mu^3 : \lambda^2\nu.$$

Or, the same result may be accomplished by replacing λ, μ and ν respectively by x, y and s , which substitutions satisfy the proportion

$$x : y : z : s = \lambda : \mu : \mu^2/\lambda^2 : \mu : \nu.$$

Following the latter substitutions, any curve (p, q) , given by the equation $\phi = 0$, meaning $\phi(\lambda, \mu, \nu) = 0$, is cut from Σ by a surface whose equation is $\phi(x, y, s) = 0$; the surface is, in this case, a cone with its vertex at the point given by $x = 0, y = 0, s = 0$ or $\lambda = 0, \nu = 0$, and will be designated by C . Since ϕ is of degree $p - q$ in λ, μ , the equation $\phi(x, y, s) = 0$ is of degree $p - q$ in the variables x, y ; hence the cuspidal edge of Σ occurs $p - q$ times on C as an edge of that cone, corresponding to the occurrence of the vertex of Σ as a $(p - q)$ -tuple point of the curve (p, q) . The cuspidal edge of Σ accordingly counts $2(p - q)$ times in the intersection of Σ and C , leaving a remainder of order $3p - 2(p - q) = p + 2q = m$, which remainder is the curve (p, q) . Hence, *the curve (p, q) is cut from Σ by a cone C of order p whose residual intersection with Σ is made up entirely of the cuspidal edge of Σ occurring $2(p - q)$ times.*

Let the order of the residual intersection in any case be denoted by R , while R_i shall denote that portion of the order of the residual intersection due to the intersection of sheets of surfaces regardless of contact of sheets, and R_c shall denote the remainder of the order of the residual intersection, the portion of the whole order of the residual intersection which arises solely from contact between the surfaces in question; then, evidently, $R = R_i + R_c$. Let the order of multiplicity of the cuspidal edge of Σ on the other surface under consideration be designated by f .

A sufficient condition in order that the curve (p, q) be the complete intersection of Σ and C is that R have the value zero, demanding that $2(p - q) = 0$ or $p = q$; such a curve does not pass at all through the vertex of Σ and is characterized by that fact among all the curves (p, q) given by equations of entirely general form. Thus, *any curve (p, q) having $p = q$ is the complete intersection of Σ with a cone C of order p .* Evidently no curve of this variety can be cut from Σ by a surface of order less than p . Curves of this kind will be grouped in the class designated as Class I.

The vertex of the cone C is the infinite point of the cuspidal edge of Σ ,—the cusp of the infinite cubic,—a point having a special relation to the geometry under consideration; and the question arises as to the number of times any curve (p, q) passes through this point. In order to determine this, the intersections of the curve (p, q) with the infinite cubic will be considered. The infinite cubic is given by the equation $\nu = 0$; hence, if the equation of the curve have its terms arranged according to the powers of ν , thus

$$\phi \equiv \phi_p + \nu \cdot \phi_{p-1} + \dots + \nu^{\theta} \cdot \phi_{p-\theta} + \dots + \nu^q \cdot \phi_{p-q} = 0,$$

the insertion of the value zero for the variable ν gives the equation $\phi_p = 0$, homogeneous in λ, μ , determining the p edges of Σ on which p intersections of the curve (p, q) with the infinite cubic lie; or, $\phi_p = 0$ gives p edges of Σ , on which edges infinite points of the curve in question are found; or, again, the equation $\phi_p = 0$ determines p directions in each of which the curve (p, q) meets the infinite plane. In the general case λ will not be a factor of ϕ_p , and, consequently, none of these p points will lie at the cusp of the infinite cubic. From the form of the equation $\phi(x, y, s) = 0$, which gives the cone C , it is evident not only that the cone C contains the cuspidal edge of Σ $p - q$ times, but also that the point in question is of multiplicity of order q on C in addition to the

multiplicity due to the multiple edge; this leads to the occurrence of the vertex of the cone C as a $2q$ -tuple point in the intersection of Σ with C , in addition to the occurrence of the cuspidal edge of Σ $2(p - q)$ times in that intersection. Hence, *the curve (p, q) passes $2q$ times through the cusp of the infinite cubic.* This makes the infinite plane meet the curve (p, q) in these $2q$ points, besides the p points elsewhere on the infinite cubic, giving in all $p + 2q$ points of intersection, —the order m of the curve in question. The only curves (p, q) , given by equations of general form, which do not pass at all through the infinite point of the cuspidal edge of Σ , are, then, those having $q = 0$, i. e., edges of the cone Σ ; other curves (p, q) having this property, but given by equations of special form, will be considered later.

Any curve (p, q) has then, in general, $p - q + 2q$ or $p + q$ points at the vertices of the two cones Σ and C ; and, since a plane through these two points contains the cuspidal and some other edge of Σ , the latter edge containing q points of (p, q) , it follows that the cuspidal edge of Σ contains, in general, no points of the curve (p, q) apart from those at the vertices of Σ and C . Hence, *the equation $\phi = 0$, in its most general form, can represent only those special curves (p, q) which cut the cuspidal edge of Σ at its infinite point and at the vertex of Σ or at either of these two points.* Thus the curves of Class I can meet the cuspidal edge of Σ only at the cusp of the infinite cubic, and must each have a $2p$ -tuple point there. The particular forms of the equation $\phi = 0$, which give the most general irreducible curves on Σ , are for the present neglected.

Let ρ denote the number of times the curve (p, q) passes through the cusp of the infinite cubic; then, in the general case, as has been seen, $\rho = 2q$.

The cones Σ and C can have no other common edge than the cuspidal edge of Σ , for the residual intersection is entirely made up of the cuspidal edge of Σ ; and it is supposed, in general, that $m \geq 2$. Any plane through the cuspidal edge of Σ contains also a second edge of Σ , and on that second edge lie q points of the curve (p, q) ; hence, through each one of these q points of (p, q) must pass an edge of the cone C ; these q edges will be, in general, distinct, and will contain no other points of (p, q) apart from the vertex of C . If Q denote the number of points in which any edge of C meets the curve (p, q) apart from the vertex of C , and P , after the analogy of p , be defined by the equation $P + 2Q = m$, then will it be possible to call the (p, q) on Σ a (P, Q) on C . But it has been seen already that $Q = 1$, hence $P = m - 2Q = m - 2 = p + 2q - 2$. The curve

(P, Q) passes through the vertex of C $2q$ times and through the vertex of Σ $p - q$ times. The cuspidal edge of Σ is a $(p - q)$ -tuple edge of C , hence any plane through it contains $p - (p - q) = q$ other edges of C ; on each of these q edges lies a point of the curve (P, Q) ; these q points, together with the $p + q$ points on the $(p - q)$ -tuple edge, make up the $p + 2q$ points in which the curve (P, Q) or the curve (p, q) meets the plane in question.

Every edge of Σ meets the curve (p, q) $p - q$ times at the vertex of Σ , but only those edges will be regarded as meeting the curve there which are given by the equation $\phi_{p-q} = 0$, where ϕ_{p-q} is the coefficient of the highest power of v occurring in the equation of the curve (p, q) ; $\phi_{p-q} = 0$ is homogeneous of degree $p - q$ in λ, μ and gives the $p - q$ edges of Σ to which the curve (p, q) is tangent at the vertex of Σ , as will be shown in a subsequent section. The curve (p, q) can then have the direction of the cuspidal edge at the vertex of Σ only if ϕ_{p-q} has a factor λ , and the direction of the inflectional edge only when μ is a factor of the same polynomial; these cases have thus far been excluded from consideration.

This completes the discussion of the curve (p, q) as cut from Σ by the cone C , and the single class of curves of complete intersection here found may now be characterized thus:

$$\text{I. } p = q, m = 3p, \pi = 0, \rho = 2q = 2p, m' = p, f = R = R_i = R_c = 0.$$

Any term in the equation $\phi(x, y, z) = 0$, which contains y^3 , may have that factor changed to x^2z as many times as it occurs, since $\Sigma \equiv y^3 - x^2z = 0$; and the surface S , whose equation is $C \equiv C' + C''y^3 = 0$, has the same intersection with Σ as the surface whose equation is obtained from $C = 0$ by adding any multiple of Σ thus:

$$C - C''\Sigma = C' + C''y^3 - C''\Sigma = C' + C''x^2z = 0.$$

This substitution of x^2z for y^3 , wherever the latter occurs, will give at once the equation of a surface S which is not in general a cone, although it still cuts the curve (p, q) from Σ and has its residual intersection with Σ made up entirely of the cuspidal edge of Σ ; and, further, it will make possible in many cases the reduction of the equation of that surface S by a factor x^a , giving a surface, not a

cone, of order less by α than p , i. e., giving a surface which is of the order $m' = p - \alpha$. The residual intersection must consist in this case of the cuspidal edge of Σ occurring to the number of $2(p - q) - 3\alpha$ times. If the factor x^α has $3\alpha = 2(p - q)$, then will the curve (p, q) in question be a curve of complete intersection; this demands that $p - q \equiv \theta \pmod{3}$, and thus it is seen that here *a curve of complete intersection passes $3k$ times through the vertex of Σ , k being an integer as great as zero.* The form which the equation $\phi = 0$ must have, in order that it be factorable thus after the substitution is performed, is found in the following manner: Let the equation $\phi = 0$ be arranged according to the powers of the variable v and represented thus:

$$\phi \equiv \phi_p + v \cdot \phi_{p-1} + v^2 \cdot \phi_{p-2} + \dots + v^{q-\gamma} \cdot \phi_{p-q+\gamma} + \dots + v^q \cdot \phi_{p-q} = 0,$$

and, further, let the general term of the polynomial $\phi_{p-q+\gamma}$ be denoted by $\alpha_{\beta, \gamma} \lambda^\beta \mu^{p-q+\gamma-\beta}$. Inserting x and y in place of λ and μ respectively in this general term, there results the form $\alpha_{\beta, \gamma} x^\beta y^{p-q+\gamma-\beta}$, which, on substituting $x^2 z$ for y^3 as many times as possible, reduces to

$$\begin{aligned} & \alpha_{\beta, \gamma} x^{\frac{1}{3}\beta + \frac{2}{3}(p-q+\gamma)} z^{\frac{2}{3}(p-q+\gamma-\beta)}, & \text{if } p-q+\gamma-\beta \equiv 0 \pmod{3}, \\ \text{or } & \alpha_{\beta, \gamma} x^{\frac{1}{3}\beta + \frac{2}{3}(p-q+\gamma-1)} y z^{\frac{2}{3}(p-q+\gamma-\beta-1)}, & \text{if } p-q+\gamma-\beta \equiv 1 \pmod{3}, \\ \text{or } & \alpha_{\beta, \gamma} x^{\frac{1}{3}\beta + \frac{2}{3}(p-q+\gamma-2)} y^2 z^{\frac{2}{3}(p-q+\gamma-\beta-2)}, & \text{if } p-q+\gamma-\beta \equiv 2 \pmod{3}, \end{aligned}$$

three cases, according to the congruence of the index of the power of μ to the modulus 3. That a factor x^α , where $\alpha = \frac{2}{3}(p - q)$ be removable by division from each of these terms, it is necessary that

$$\begin{aligned} \frac{1}{3}\beta + \frac{2}{3}(p-q+\gamma) &\geq \frac{2}{3}(p-q), \text{ i. e., } \beta \geq -2\gamma, & \text{if } p-q+\gamma-\beta \equiv 0 \pmod{3} \\ \frac{1}{3}\beta + \frac{2}{3}(p-q+\gamma-1) &\geq \frac{2}{3}(p-q), \text{ i. e., } \beta \geq 2-2\gamma, & \text{if } p-q+\gamma-\beta \equiv 1 \pmod{3}, \\ \frac{1}{3}\beta + \frac{2}{3}(p-q+\gamma-2) &\geq \frac{2}{3}(p-q), \text{ i. e., } \beta \geq 4-2\gamma, & \text{if } p-q+\gamma-\beta \equiv 2 \pmod{3}. \end{aligned}$$

If $\gamma = 0$, the last terms of ϕ_{p-q} , on which terms alone of this group any restriction could be placed by these conditions, are of the form

$$\alpha_{3,0} \lambda^3 \mu^{p-q-3} + \alpha_{2,0} \lambda^2 \mu^{p-q-2} + \alpha_{1,0} \lambda \mu^{p-q-1} + \alpha_{0,0} \mu^{p-q},$$

and, since $p - q \equiv 0 \pmod{3}$, the necessary conditions for β are all satisfied if only $\alpha_{1,0}$ vanish. Similarly, it is seen that no restrictions whatever are imposed

by these conditions for β on any of the terms in the groups where $\gamma \geq 1$. Hence, there is given a curve of complete intersection whenever $p - q \equiv 0 \pmod{3}$, and $a_{1,0} = 0$. This case, which includes the curves of Class I above (the condition $a_{1,0} = 0$ being always satisfied there), may be called Class I', and is characterized thus:

$$\text{I'. } p > q, p - q \equiv 0 \pmod{3}, a_{1,0} = 0, m = p + 2q, \pi = p - q, \\ \rho = 2q, m' = \frac{1}{3}(p + 2q), f = R = R_i = R_c = 0.$$

In general it is not possible to divide out from the equation of the surfaces S a factor of degree as great as the order of the residual intersection, but a certain reduction can usually be made. The nature of such reduction in the general case will now be considered. The equation $\phi = 0$, when arranged according to the ascending powers of ν , takes the form

$$\phi \equiv \phi_p + \nu \cdot \phi_{p-1} + \nu^2 \cdot \phi_{p-2} + \dots + \nu^\theta \cdot \phi_{p-\theta} + \dots + \nu^q \cdot \phi_{p-q} = 0;$$

here arise at once three cases, according as

$$\text{or } \left. \begin{array}{l} 1) p - q \equiv 0 \pmod{3}, \\ 2) p - q \equiv 1 \pmod{3}, \\ 3) p - q \equiv 2 \pmod{3}; \end{array} \right\} \text{i. e., according as } \left\{ \begin{array}{ll} 1) p - q = 3\Delta, & \\ 2) p - q = 3\Delta + 1, & \\ \text{or } 3) p - q = 3\Delta + 2. & \end{array} \right.$$

where Δ is an integer as great as zero.

Case I. $p - q = 3\Delta$. Here Δ may be supposed to be as great as unity, since the case where $p - q = 0$ has been already considered. The general term of the polynomial ϕ_{p-q} is of the form $a_r \lambda^{3\Delta-r} \mu^r$; this term becomes at once $a_r x^{3\Delta-r} y^r$, and this, after neglecting the coefficient and changing y^3 to $x^2 z$ as many times as possible, may be reduced to one of three forms, according to the congruence of r to the modulus 3, thus:

$$\begin{aligned} x^{3\Delta-r} y^r &= x^{3\Delta-r+\frac{2}{3}r} z^{\frac{1}{3}r} \equiv x^{3\Delta-\frac{1}{3}r} z^{\frac{1}{3}r}, & \text{if } r \equiv 0 \pmod{3}; \\ &= x^{3\Delta-r+\frac{2}{3}(r-1)} y z^{\frac{1}{3}(r-1)} \equiv x^{3\Delta-\frac{1}{3}(r+2)} y z^{\frac{1}{3}(r-1)}, & \text{if } r \equiv 1 \pmod{3}; \\ &= x^{3\Delta-r+\frac{2}{3}(r-2)} y^2 z^{\frac{1}{3}(r-2)} \equiv x^{3\Delta-\frac{1}{3}(r+4)} y^2 z^{\frac{1}{3}(r-2)}, & \text{if } r \equiv 2 \pmod{3}. \end{aligned}$$

In each case there occurs a factor x^α , where $\alpha \geq 3\Delta - \frac{1}{3}(r+4) \geq \frac{2}{3}(p-q) - 1$, since $\Delta = \frac{1}{3}(p-q)$ and $r \leq p-q$; hence division by $x^{\frac{2}{3}(p-q)-1}$ is made possible in all the terms obtained from the group ϕ_{p-q} . Since the terms derived from any other group $\phi_{p-\theta}$, where $\theta \leq q-1$, will be at least one degree higher in the variables x, y than those coming from ϕ_{p-q} , it is evident that the factor

$x^{\frac{2}{3}(p-q)-1}$ can be rejected from the entire equation. The order of the surface S then becomes $m' = p - \frac{2}{3}(p-q) + 1 = \frac{1}{3}(p+2q) + 1$, and the residual intersection must consist of the cuspidal edge of Σ occurring $2(p-q) - 3\alpha = 2(p-q) - 2(p-q) + 3 = 3$ times, requiring that $f=1$, $R_i=2$ and $R_e=1$; hence the surface S has the plane given by $x=0$ for its tangent plane all along the cuspidal edge of Σ , the surfaces S and Σ having contact all along that line. The same may be found at once to be true by investigating somewhat more carefully the forms of the terms given above, thus: Dividing out $x^a \equiv x^{\frac{2}{3}(p-q)-1}$ from each of the three forms given leaves

$$\begin{aligned} x^{p-q-\frac{1}{3}r-\frac{2}{3}(p-q)+1} z^{\frac{1}{3}r} &\equiv x^{\frac{1}{3}(p-q)-\frac{1}{3}r+1} z^{\frac{1}{3}r}, & \text{if } r \equiv 0 \pmod{3}, \\ x^{p-q-\frac{1}{3}(r+2)-\frac{2}{3}(p-q)+1} y z^{\frac{1}{3}(r-1)} &\equiv x^{\frac{1}{3}(p-q)-\frac{1}{3}(r-1)} y z^{\frac{1}{3}(r-1)}, & \text{if } r \equiv 1 \pmod{3}, \\ \text{or } x^{p-q-\frac{1}{3}(r+4)-\frac{2}{3}(p-q)+1} y^2 z^{\frac{1}{3}(r-2)} &\equiv x^{\frac{1}{3}(p-q)-\frac{1}{3}(r+1)} y^2 z^{\frac{1}{3}(r-2)}, & \text{if } r \equiv 2 \pmod{3}. \end{aligned}$$

Since $r \leq p-q$, the lowest terms in x, y come from the first group and are of the first power in those variables, containing only x ; all terms from the second and third groups will be of at least the second degree in x, y and will contain y . Curves of the kind here considered will be said to belong to Class II and may be characterized thus:

$$\begin{aligned} \text{II. } p > q, \quad p-q &\equiv 0 \pmod{3}, \quad m = p+2q, \quad \pi = p-q, \quad \rho = 2q, \\ m' &= \frac{1}{3}(p+2q) + 1, \quad f = 1, \quad R = 3, \quad R_i = 2, \quad R_e = 1. \end{aligned}$$

Evidently no curves of complete intersection are included in this class.

Case 2. $p-q = 3\Delta + 1$. The general term of ϕ_{p-q} , of the form $\alpha_r \lambda^{3\Delta-r+1} \mu^r$, becomes at once $\alpha_r x^{3\Delta-r+1} y^r$; and this, after neglecting the coefficient and changing y^3 to x^2z as many times as possible, takes one of three forms,

$$\begin{aligned} x^{3\Delta-r+1} y^r &= x^{3\Delta-r+1+\frac{2}{3}r} z^{\frac{1}{3}r} \equiv x^{3\Delta-\frac{1}{3}r+1} z^{\frac{1}{3}r}, & \text{if } r \equiv 0 \pmod{3}, \\ &= x^{3\Delta-r+1+\frac{2}{3}(r-1)} y z^{\frac{1}{3}(r-1)} \equiv x^{3\Delta-\frac{1}{3}(r-1)} y z^{\frac{1}{3}(r-1)}, & \text{if } r \equiv 1 \pmod{3}, \\ &= x^{3\Delta-r+1+\frac{2}{3}(r-2)} y^2 z^{\frac{1}{3}(r-2)} \equiv x^{3\Delta-\frac{1}{3}(r+1)} y^2 z^{\frac{1}{3}(r-1)}, & \text{if } r \equiv 2 \pmod{3}. \end{aligned}$$

These terms are all divisible by x^a , where $a = 3\Delta - \frac{1}{3}(r+1) \geq \frac{2}{3}(p-q-1)$. It is evident that all the terms of higher degrees in x, y coming from the groups $\phi_{p-\theta}$, where $\theta \leq q-1$ can be divided by the same x^a ; hence, the entire equation contains the factor $x^{\frac{2}{3}(p-q)-1}$. The order of the surface S is then $p-\alpha = p - \frac{2}{3}(p-q-1) = \frac{1}{3}(p+2q+2)$, and the residual intersection consists of the cuspidal edge of Σ occurring $2(p-q) - 3\alpha = 2(p-q) - 2(p-q-1) = 2$ times, demanding that $f=1$, $R=R_i=2$ and $R_e=0$; thus the surface S con-

tains the cuspidal edge of Σ once. The same may be seen at once to be true from the forms of the terms given above. Curves of this kind will be grouped in Class III and may be described thus:

$$\text{III. } p - q \equiv 1 \pmod{3}, \quad m = p + 2q, \quad \pi = p - q, \quad \rho = 2q, \\ m' = \frac{1}{3}(p + 2q + 2), \quad f = 1, \quad R = 2, \quad R_i = 2, \quad R_c = 0.$$

Case 3. $p - q = 3\Delta + 2$. The general term of Φ_{p-q} is of the form $a_r \lambda^{3\Delta - r + 2} \mu^r$; this becomes at once $a_r x^{3\Delta - r + 2} y^r$; and this, after neglecting the coefficient a_r and replacing y^3 by $x^2 z$ as many times as possible, takes one of three forms, thus:

$$\begin{aligned} x^{3\Delta - r + 2} y^r &= x^{3\Delta - r + 2 + \frac{2}{3}r} z^{\frac{1}{3}r} \equiv x^{3\Delta - \frac{1}{3}r + 2} z^{\frac{1}{3}r}, & \text{if } r \equiv 0 \pmod{3}, \\ &= x^{3\Delta - r + 2 + \frac{2}{3}(r-1)} y z^{\frac{1}{3}(r-1)} \equiv x^{3\Delta - \frac{1}{3}(r-4)} y z^{\frac{1}{3}(r-1)}, & \text{if } r \equiv 1 \pmod{3}, \\ &= x^{3\Delta - r + 2 + \frac{2}{3}(r-2)} y^2 z^{\frac{1}{3}(r-2)} \equiv x^{3\Delta - \frac{1}{3}(r-2)} y^2 z^{\frac{1}{3}(r-2)}, & \text{if } r \equiv 2 \pmod{3}. \end{aligned}$$

These terms are all divisible by x^α , where $\alpha = 3\Delta - \frac{1}{3}(r-2) \geq \frac{2}{3}(p-q-2)$. It is evident that all the terms of higher degrees in x, y , coming from the groups $\Phi_{p-\theta}$, where $\theta \leq q-1$, can be divided by the same x^α , hence $x^{\frac{2}{3}(p-q-2)}$ is a factor of the entire equation. The order of the surface S may be reduced thus to $p - \alpha = p - \frac{2}{3}(p-q-2) = \frac{1}{3}(p+2q+4)$, and the residual intersection consists of the cuspidal edge of Σ occurring $2(p-q) - 3\alpha = 2(p-q) - 2(p-q-2) = 4$ times, giving $R = 4$; and, since each term above is of degree at least as great as two in the variables x, y , after the rejection of the factor x^α , it is clear that $f = 2$, $R_i = 4$, and, consequently, $R_c = 0$; accordingly, the surface S contains the cuspidal edge of Σ twice. This case includes the curves which will be said to form Class IV, and may be characterized thus:

$$\text{IV. } p - q \equiv 2 \pmod{3}, \quad m = p + 2q, \quad \pi = p - q, \quad \rho = 2q, \\ m' = \frac{1}{3}(p + 2q + 4), \quad f = 2, \quad R = 4, \quad R_i = 4, \quad R_c = 0.$$

Since $p - q \equiv p + 2q \pmod{3}$, and since a necessary condition for a curve (p, q) of complete intersection is that $m = p + 2q \equiv 0 \pmod{3}$, it follows that no curves of total intersection are included in Classes III and IV.

This disposes of the cases of all curves (p, q) which are represented by equations of the most general form for the values of p and q in question. But, since every such general equation has been found to represent a curve (p, q) passing $p - q$ times through the vertex of Σ and $2q$ times through the cusp of the infinite cubic, it is evident that many general curves, including all curves of

order greater than unity which do not contain either the vertex of Σ or the cusp of the infinite cubic, remain to be given by equations of special forms.

Curves (p, q) given by Equations of Special Forms.

It is possible to determine the conditions under which a higher power of x than that given above can be rejected from the equation obtained from the performance of the required substitutions in the equation $\phi = 0$; such greater reduction will give a surface of lower order containing the curve in question, and it may happen that the curve itself will suffer a reduction in its order by the rejection of some of the terms which appear in the general equation for its arrangement of p and q .

If the terms of the general equation of the curve (p, q) are arranged according to powers of v , thus:

$$\phi \equiv \phi_p + \phi_{p-1} \cdot v + \phi_{p-2} \cdot v^2 + \dots + \phi_{p-q+\gamma} \cdot v^{q-\gamma} + \dots + \phi_{p-q} \cdot v^q = 0,$$

any term of the general group $\phi_{p-q+\gamma}$ is of the form $\alpha_{\beta,\gamma} \lambda^\beta \mu^{p-q+\gamma-\beta}$ and, if the substitution be performed, this term becomes at once $\alpha_{\beta,\gamma} x^\beta y^{p-q+\gamma-\beta}$; and this, on changing y^3 to $x^2 z$ as many times as possible, takes the form

$$\begin{aligned} & \alpha_{\beta,\gamma} x^{\frac{1}{3}\beta + \frac{2}{3}(p-q+\gamma)} z^{\frac{1}{3}(p-q+\gamma-\beta)}, & \text{if } p-q+\gamma-\beta \equiv 0 \pmod{3}, \\ \text{or } & \alpha_{\beta,\gamma} x^{\frac{1}{3}\beta + \frac{2}{3}(p-q+\gamma-1)} y z^{\frac{1}{3}(p-q+\gamma-\beta-1)}, & \text{if } p-q+\gamma-\beta \equiv 1 \pmod{3}, \\ \text{or } & \alpha_{\beta,\gamma} x^{\frac{1}{3}\beta + \frac{2}{3}(p-q+\gamma-2)} y^2 z^{\frac{1}{3}(p-q+\gamma-\beta-2)}, & \text{if } p-q+\gamma-\beta \equiv 2 \pmod{3}. \end{aligned}$$

If $p-q \equiv 0 \pmod{3}$, it has been found that the factor x^α , where $\alpha = \frac{2}{3}(p-q) - 1$, can always be removed by division from the substituted equation; if a factor $x^{\alpha+\alpha'}$, where $\alpha' \geq 1$, is to be removable in the same way, it is necessary that the exponent of x in each of the above reduced terms be as great as $\alpha + \alpha' = \frac{2}{3}(p-q) - 1 + \alpha'$; i. e., it is necessary that $\frac{1}{3}\beta + \frac{2}{3}(p-q+\gamma) \geq \frac{2}{3}(p-q) - 1 + \alpha'$, which requires that

$$\beta \geq 3\alpha' - 2\gamma - 3, \text{ if } p-q+\gamma-\beta \equiv 0 \pmod{3}.$$

Similarly must $\beta \geq 3\alpha' - 2\gamma - 1$, if $p-q+\gamma-\beta \equiv 1 \pmod{3}$,

and $\beta \geq 3\alpha' - 2\gamma + 1$, if $p-q+\gamma-\beta \equiv 2 \pmod{3}$.

These conditions require in general that all those terms of the general equation of (p, q) vanish, where β , or better $\beta + 2\gamma$, falls below a definite limit depending on α' . For smaller values of α' , the terms which must vanish in any group

$\phi_{p-q+\gamma}$ are those of lower degrees in λ , and the larger number of terms must vanish, in general, in those groups where γ has the smaller values, i. e., where ν enters to its higher powers. The terms in the group $\phi_{p-q} \cdot \nu^q$ cannot all be made to vanish, else the species of the curve in question would be changed by a reduction in the value of q ; and the highest value of α' may occur only when the single term highest in $\beta + 2\gamma$ in this group, viz., $a_{p-q,0} \lambda^{p-q} \nu^q$ alone remains; this term, under the required substitution, becomes $a_{p-q,0} x^{p-q} s^q$ and the upper limit for α' here involved is found to be $\frac{1}{3}(p-q) + 1$, since $\alpha + \alpha' = \frac{2}{3}(p-q) - 1 + \alpha' \geq p - q$.

Again, there must occur in the equation of the curve in question one term at least which does not involve the variable λ , else that equation would be reducible by some power of that variable. Among the possible terms in μ, ν , that involving the highest value of $\beta + 2\gamma$, and hence the highest value of α' , is the term $a_{0,q} \mu^p$; this term becomes, under the required substitution, $a_{0,q} y^p$, and, on changing y^3 to $x^2 z$ as many times as possible, it is found that

$$\begin{aligned} a_{0,q} y^p &= a_{0,q} x^{\frac{2}{3}p} z^{\frac{1}{3}p}, & \text{if } p \equiv 0 \pmod{3}, \\ \text{or} &= a_{0,q} x^{\frac{2}{3}(p-1)} y z^{\frac{1}{3}(p-1)}, & \text{if } p \equiv 1 \pmod{3}, \\ \text{or} &= a_{0,q} x^{\frac{2}{3}(p-2)} y^2 z^{\frac{1}{3}(p-2)}, & \text{if } p \equiv 2 \pmod{3}. \end{aligned}$$

Hence $\alpha + \alpha'$ has for an upper limit $\frac{2}{3}p$, $\frac{2}{3}(p-1)$ or $\frac{2}{3}(p-2)$ according as $p \equiv 0, 1$ or $2 \pmod{3}$; since $\alpha = \frac{2}{3}(p-q) - 1$, this gives upper limits for α' as follows:

$$\begin{aligned} \alpha' &\leq \frac{2}{3}q + 1, & \text{if } p \equiv 0 \pmod{3}, \\ \text{or} &\alpha' \leq \frac{2}{3}q + \frac{1}{3}, & \text{if } p \equiv 1 \pmod{3}, \\ \text{or} &\alpha' \leq \frac{2}{3}q - \frac{1}{3}, & \text{if } p \equiv 2 \pmod{3}. \end{aligned}$$

Then not only must α' be less than or at most equal to $\frac{1}{3}(p-q) + 1$, but it must also satisfy the appropriate one of the last three conditions here found.

To restrict the general equation of (p, q) in such a way that the desired reduction by $x^{\alpha+\alpha'}$ may be made, the coefficient $a_{\beta,\gamma}$ in the assumed general equation must take the value zero whenever the conditions found above for β are not fulfilled. Examining these conditions, it appears that $a_{\beta,\gamma}$ would vanish by the third condition above, if the equations $\beta = 3\alpha' - 2\gamma$ and $p - q + \gamma - \beta \equiv 2 \pmod{3}$ were both satisfied, since $\beta = 3\alpha' - 2\gamma < 3\alpha' - 2\gamma + 1$; but if $\beta = 3\alpha' - 2\gamma$, then is $p - q + \gamma - \beta = p - q + 3(\gamma - \alpha') \equiv 0 \pmod{3}$, the condition that $p - q + \gamma - \beta \equiv 2 \pmod{3}$ is not satisfied, and $a_{\beta,\gamma}$ does not

vanish in this case. If $\beta = 3\alpha' - 2\gamma - 2$, then $p - q + \gamma - \beta = p - q + 3(\gamma - \alpha') + 2 \equiv 2 \pmod{3}$, and the coefficient $a_{\beta, \gamma}$ vanishes in this case under the third condition above, since $3\alpha' - 2\gamma - 2 < 3\alpha' - 2\gamma + 1$. Also $a_{\beta, \gamma}$ must vanish for all values of β, γ where $0 \leq \beta \leq 3\alpha' - 2\gamma - 4$.

In order that all these conditions may be applicable to the coefficients of the equation in question, it is necessary that in the case of no one of them shall values of β, γ occur, which are not found in the coefficient $a_{\beta, \gamma}$ of some term of the equation; i. e., for all values of γ occurring, must $3\alpha' - 2\gamma - 2 \leq p - q + \gamma$; this reduces to the condition that $\alpha' \leq \frac{1}{3}(p - q) + \frac{2}{3} + \gamma$; difficulty arises here only when α' takes its largest value $\frac{1}{3}(p - q) + 1$, and γ has the value zero; in that case $\frac{1}{3}(p - q) + 1 > \frac{1}{3}(p - q) + \frac{2}{3} + \gamma$, and this condition fails to find application; the form of the coefficient for the application of this condition in this case is $a_{\beta, \gamma} = a_{3\alpha' - 2\gamma - 2, 0} = a_{p - q + 1, 0}$, and it is evident that the equation of the curve (p, q) contains no such coefficient. If, then, $\alpha' = \frac{1}{3}(p - q) + 1$, one condition above is inapplicable; but if $\alpha' < \frac{1}{3}(p - q) + 1$, the conditions all find application to the coefficients of the given equation.

The number of conditions imposed on the equation $\phi = 0$ in the case where $p - q \equiv 0 \pmod{3}$ is then $\sum_0^{[\frac{1}{3}\alpha' - 1]} (3\alpha' - 2\gamma - 2) + 1$, if $\alpha' < \frac{1}{3}(p - q) + 1$ and even, or $\sum_0^{[\frac{1}{3}\alpha' - 1]} (3\alpha' - 2\gamma - 2)$, if $\alpha' < \frac{1}{3}(p - q) + 1$ and odd; this amounts to $\frac{3}{4}\alpha'(3\alpha' - 2) + 1$, if α' be even, and to $\frac{1}{4}(3\alpha' - 1)^2$, if α' be odd; hence the number of conditions imposed on the general equation of (p, q) to ensure the desired reduction by $x^{a+\alpha'}$, when $\alpha' = \frac{1}{3}(p - q) + 1$, is $\frac{3}{4}\alpha'(3\alpha' - 2)$, if α' be even, and $\frac{1}{4}(3\alpha' - 1)^2 - 1$, if α' be odd and greater than unity, but unity, if $\alpha' = 1$.

The equation $\phi = 0$, after such conditions have been imposed on its coefficients, will be called the restricted equation $\phi = 0$; the curve represented by an equation thus restricted will be designated as a $(p, q)_{\alpha'}$ and the surface cutting such a curve $(p, q)_{\alpha'}$ from Σ will be denoted by S' .

The most general equation for a curve (p, q) , where $p - q \equiv 0 \pmod{3}$, has been found to give a curve of order $p + 2q$, which is cut from Σ by a surface S of order $\frac{1}{3}(p + 2q) + 1$, the residual intersection consisting of the cuspidal edge of Σ occurring three times, so that $f = 1$, $R = 3$, $R_i = 2$, $R_c = 1$. If the condition that $\alpha' = 1$ be imposed, the surface S' will be given by an equation of

degree $\frac{1}{3}(p+2q)$. The conditions imposed on the equation $\phi=0$ are not in general such that the restricted equation $\phi=0$ will be a reducible equation; hence, if the curve (p, q) be broken up at all to form the curve $(p, q)_{\alpha'}$, it can be brought about only by introducing into the locus of (p, q) the cuspidal edge of Σ as many times as is demanded by the value of α' , that value having been reduced first by the order of the residual intersection of S' and Σ ; hence it will in general be true that the residual intersection of S' and Σ , if any such residual intersection exist, consists entirely of the cuspidal edge of Σ occurring as many times as the order of the residual intersection in question. This will be true generally, whether $p-q \equiv 0 \pmod{3}$, or not.

In the case under consideration, where $p-q \equiv 0 \pmod{3}$, the condition that $\alpha'=1$ does not cause the coefficient $a_{0,0}$ to vanish, for $\beta=\gamma=0$ makes $p-q+\gamma-\beta=p-q \equiv 0 \pmod{3}$, and the first condition for β , viz., that $\beta \geq 3\alpha'-2\gamma-3$, is satisfied here since $0=3\alpha'-3$; hence the term $a_{0,0}\mu^{p-q}\nu^q$ is not removed from the equation of (p, q) ; this term, under the substitutions required, becomes $a_{0,0}y^{p-q}s^q = a_{0,0}x^{\frac{2}{3}(p-q)}z^{\frac{1}{3}(p-q)}s^q$; and the rejection of the factor $x^{\alpha+\alpha'}=x^{\frac{2}{3}(p-q)}$ gives $a_{0,0}z^{\frac{1}{3}(p-q)}s^q$, a term free from both x and y ; consequently the cuspidal edge of Σ does not occur at all on S' and the curve $(p, q)_{\alpha'}$ is accordingly a curve of total intersection of S' and Σ whenever $\alpha'=1$. The number of conditions imposed on the equation in this case is found by the formula given to be $\frac{1}{4}(3\alpha'-1)^2 = \frac{1}{4}(3-1)^2 = 1$, if $\alpha' < \frac{1}{3}(p-q)+1$, i. e. if $p > q$, which is always true in the cases under consideration, the case where $p=q$ having been finally disposed of on page 277; the single condition to be applied here, as determined by the value $\beta \geq 3\alpha'-2\gamma-2$ and $p-q+\gamma-\beta \equiv 2 \pmod{3}$, is $a_{\beta,\gamma} = a_{3\alpha'-2\gamma-2,\gamma} = a_{1-2\gamma,\gamma} = a_{1,0} = 0$.

In the case where α' had the value zero, the residual intersection was found to be made up of the cuspidal edge of Σ occurring three times. The rejection of $x^{\alpha'}=x$, since the plane $x=0$ contains that edge three times, simply removes the residual intersection and does not change the order of multiplicity of the vertex of Σ or of the cusp of the infinite cubic on the curve proper; i. e., π and ρ have the same values in the cases of the curve $(p, q)_1$ and the curve (p, q) ; viz., $\pi=p-q$ and $\rho=2q$.

The curves $(p, q)_1$, where $p-q \equiv 0 \pmod{3}$, are then identical with those of Class I' (page 279), characterized thus,

$$\begin{aligned} \text{I'. } p > q, p-q \equiv 0 \pmod{3}, \alpha' = 1, a_{1,0} = 0, m = p+2q, \pi = p-q, \\ \rho = 2q, m' = \frac{1}{3}(p+2q), f = R = R_i = R_c = 0. \end{aligned}$$

If $\alpha' \geq 2$, the natures of the curve $(p, q)_{\alpha'}$ resulting, and of the surface S' cutting the same from Σ , are found as follows. The order of the surface S' is $\frac{1}{3}(p + 2q) + 1$; hence the order of the surface S' must be less by α' than that number, i. e., $\frac{1}{3}(p + 2q) + 1 - \alpha'$. It has been seen that the coefficient $\alpha_{\beta, \gamma}$, where $\beta = 3\alpha' - 2\gamma - 3$, is not made to vanish for any value of α' ; since, in any group $\phi_{p-q+\gamma}$, the variable λ occurs in general in some term to the power $p - q + \gamma$, there will occur in the equation of the curve $(p, q)_{\alpha'}$ a term where $\beta = 3\alpha' - 2\gamma - 3$ if $0 \leq 3\alpha' - 2\gamma - 3 \leq p - q + \gamma$; the former condition is satisfied when $\gamma = 0$ and $\alpha' \geq 2$; the latter condition demands that $3\alpha' \leq p - q + 3\gamma + 3$ and is always fulfilled, since $\alpha' \leq \frac{1}{3}(p - q) + 1$; hence the equation of $(p, q)_{\alpha'}$ always contains a term of the form

$$\alpha_{\beta, \gamma} \lambda^{\beta} \mu^{p-q+\gamma-\beta} \nu^{q-\gamma} = \alpha_{3\alpha'-2\gamma-3, \gamma} \lambda^{3\alpha'-2\gamma-3} \mu^{p-q+3\gamma-3\alpha'+3} \nu^{q-\gamma},$$

which becomes, under the required substitution,

$$\alpha_{3\alpha'-2\gamma-3, \gamma} x^{3\alpha'-2\gamma-3} y^{p-q+3\gamma-3\alpha'+3} s^{q-\gamma}.$$

If in this term x^2z be substituted for y^3 as many times as is possible, it takes the form $\alpha_{3\alpha'-2\gamma-3, \gamma} x^{\frac{1}{3}(p-q)-1+\alpha'} z^{\frac{1}{3}(p-q)+\gamma-\alpha'+1} s^{q-\gamma}$. Rejecting the factor $x^{\alpha'+\alpha'} = x^{\frac{1}{3}(p-q)-1+\alpha'}$ from this term leaves only $\alpha_{3\alpha'-2\gamma-3, \gamma} z^{\frac{1}{3}(p-q)+\gamma-\alpha'+1} s^{q-\gamma}$, which is of the zeroth degree in x, y ; consequently, the equation of the surface S' , which cuts the curve $(p, q)_{\alpha'}$ from Σ , where $p - q \equiv 0 \pmod{3}$, is of the zeroth degree in the variables x, y whenever $\alpha' \geq 2$. The residual intersection, consisting entirely in every case of the cuspidal edge of Σ occurring a certain number of times, is wanting, since the surface S' does not contain that edge at all. But the rejection of the factor $x^{\alpha'}$ from the equation of the surface S' has reduced the common intersection of the two surfaces by the cuspidal edge counting $3\alpha'$ times, and has consequently not only removed the former residual intersection of order three, but has also subjected the curve to a reduction of its order by $3(\alpha' - 1)$, since the cuspidal edge has been rejected that number of times from the curve portion of the intersection. Hence, in this case, where $p - q \equiv 0 \pmod{3}$, the restricted equation $\phi = 0$ represents a curve $(p, q)_{\alpha'}$ of order $m = p + 2q - 3(\alpha' - 1)$; and this curve $(p, q)_{\alpha'}$ is a curve of total intersection of the newly found surface S' with Σ .

The rejection of the cuspidal edge of Σ thus $3(\alpha' - 1)$ times from the curve (p, q) effects a reduction by that number in the order of multiplicity of the vertex of Σ as a point of the curve; in the case of the curve (p, q) it has been

found that $\pi = p - q$; hence, for the curve $(p, q)_{\alpha'}$, $\pi = p - q - 3(\alpha' - 1)$. Similarly, the curve $(p, q)_{\alpha'}$ passes $3(\alpha' - 1)$ less times through the cusp of the infinite cubic than does the curve (p, q) ; consequently, $\rho = 2q - 3(\alpha' - 1)$. Since neither π nor ρ can be negative, the upper limits found already for α' are here made evident again; and here, as in all other cases of curves of total intersection thus far considered, $\pi \equiv 0 \pmod{3}$. If both π and ρ have in any case the value zero, then must the curve in question be of a more general character than the (p, q) of the same order, which is known to pass $p - q$ times through the vertex of Σ and $2q$ times through the cusp of the infinite cubic (since both $p - q$ and $2q$ can in no case have the value zero). The curve $(p, q)_{\alpha'}$, which is, from this point of view, one of the most general curves on Σ , occurs when $\pi = p - q - 3(\alpha' - 1) = \rho = 2q - 3(\alpha' - 1) = 0$, which demands that $p = 3q$ and $\alpha' = \frac{2}{3}q + 1 = \frac{2}{3}p + 1$; while $m = p$, $m' = \frac{1}{3}p$, and $2q = p - q \equiv 0 \pmod{3}$, making $q \equiv 0 \pmod{3}$ and $p \equiv 0 \pmod{9}$. Such a curve is the $(9, 3)_3$, where $\alpha = 3$ and $\alpha' = 3$, while $m = 9$, $m' = 3$, and $\pi = \rho = 0$; this case may now be worked out according to the conditions already given, as follows: The most general equation of the curve $(9, 3)$ is of the form

$$\begin{aligned} & a_{9,3}\lambda^9 + a_{8,3}\lambda^8\mu + a_{7,3}\lambda^7\mu^2 + a_{6,3}\lambda^6\mu^3 + a_{5,3}\lambda^5\mu^4 + a_{4,3}\lambda^4\mu^5 + a_{3,3}\lambda^3\mu^6 + a_{2,3}\lambda^2\mu^7 \\ & + a_{1,3}\lambda\mu^8 + a_{0,3}\mu^9 + (a_{8,2}\lambda^8 + a_{7,2}\lambda^7\mu + a_{6,2}\lambda^6\mu^2 + a_{5,2}\lambda^5\mu^3 + a_{4,2}\lambda^4\mu^4 \\ & + a_{3,2}\lambda^3\mu^5 + a_{2,2}\lambda^2\mu^6 + a_{1,2}\lambda\mu^7 + a_{0,2}\mu^8)\nu + (a_{7,1}\lambda^7 + a_{6,1}\lambda^6\mu + a_{5,1}\lambda^5\mu^2 \\ & + a_{4,1}\lambda^4\mu^3 + a_{3,1}\lambda^3\mu^4 + a_{2,1}\lambda^2\mu^5 + a_{1,1}\lambda\mu^6 + a_{0,1}\mu^7)\nu^2 + (a_{6,0}\lambda^6 + a_{5,0}\lambda^5\mu \\ & + a_{4,0}\lambda^4\mu^2 + a_{3,0}\lambda^3\mu^3 + a_{2,0}\lambda^2\mu^4 + a_{1,0}\lambda\mu^5 + a_{0,0}\mu^6)\nu^3 = 0. \end{aligned}$$

Imposing the condition that all terms vanish where $\beta = 3\alpha' - 2\gamma - 2 = 7 - 2\gamma$, and where $\beta \leq 3\alpha' - 2\gamma - 4 = 5 - 2\gamma$, removes $\frac{1}{2}(3\alpha' - 1)^2 - 1 = 15$ terms, since it makes $a_{1,3} = a_{3,2} = a_{1,2} = a_{0,2} = a_{5,1} = a_{3,1} = a_{2,1} = a_{1,1} = a_{0,1} = a_{5,0} = a_{4,0} = a_{3,0} = a_{2,0} = a_{1,0} = a_{0,0} = 0$; this leaves the equation of the curve $(p, q)_{\alpha'}$ in the form

$$\begin{aligned} & a_{9,3}\lambda^9 + a_{8,3}\lambda^8\mu + a_{7,3}\lambda^7\mu^2 + a_{6,3}\lambda^6\mu^3 + a_{5,3}\lambda^5\mu^4 + a_{4,3}\lambda^4\mu^5 + a_{3,3}\lambda^3\mu^6 \\ & + a_{2,3}\lambda^2\mu^7 + a_{0,3}\mu^9 + (a_{8,2}\lambda^8 + a_{7,2}\lambda^7\mu + a_{6,2}\lambda^6\mu^2 + a_{5,2}\lambda^5\mu^3 + a_{4,2}\lambda^4\mu^4 \\ & + a_{2,2}\lambda^2\mu^5)\nu + (a_{7,1}\lambda^7 + a_{6,1}\lambda^6\mu + a_{4,1}\lambda^4\mu^3)\nu^2 + a_{6,0}\lambda^6\nu^3 = 0. \end{aligned}$$

Substituting x, y and s for λ, μ and ν respectively, changing y^3 to x^2z wherever possible, and removing the factor $x^{a+a'} = x^6$, there remains

$$\begin{aligned} & \alpha_{9,3} x^3 + \alpha_{8,3} x^2 y + \alpha_{7,3} x y^2 + \alpha_{6,3} x^2 z + \alpha_{5,3} x y z + \alpha_{4,3} y^2 z + \alpha_{3,3} x z^2 + \alpha_{2,3} y z^2 \\ & + \alpha_{0,3} z^3 + \alpha_{8,2} x^2 s + \alpha_{7,2} x y s + \alpha_{6,2} y^2 s + \alpha_{5,2} x z s + \alpha_{4,2} y z s + \alpha_{2,2} z^2 s \\ & + \alpha_{7,1} x s^2 + \alpha_{6,1} y s^2 + \alpha_{4,1} z s^2 + \alpha_{6,0} s^3 = 0, \end{aligned}$$

the equation of the cubic surface S' , whose intersection with Σ is the curve $(9, 3)_3$ of order $m = p + 2q - 3 (\alpha' - 1) = 9$. This equation contains all the terms of the most general homogeneous cubic equation in the four variables x, y, z, s except the term in y^3 . Suppose there had occurred the terms $\alpha'_{6,3} y^3$ and $\alpha''_{6,3} x^2 z$ in this equation, making it represent the general cubic surface; the substitution of $x^2 z$ for y^3 in the former term would not have affected the intersection of the surface with Σ , and the two terms $\alpha'_{6,3} y^3$ and $\alpha''_{6,3} x^2 z$, now $\alpha'_{6,3} x^2 z$ and $\alpha''_{6,3} x^2 z$, could have been combined as a single term $\alpha_{6,3} x^2 z$; hence, the intersection of this surface S' with Σ gives the most general curve of total intersection that can be cut from Σ by a cubic surface; such intersection is a general curve of order 9, and is deservedly ranked as a curve of more general character than those which are required to pass a certain number of times through the vertex of Σ or the cusp of the infinite cubic, or both. Similarly, the general sextic surface cuts from Σ a curve of the species $(18, 6)_6$ and of order 18, a curve of total intersection, where $\alpha = 7$ and $\alpha' = 5$. And in the same manner may be found here the curves of total intersection of Σ with the general surfaces S of orders 9, 12, 15, 18, 21, etc., those curves being of the species $(27, 9)_7, (36, 12)_8, (45, 15)_{11}$, etc.

The curves of total intersection here occurring will be grouped as Class II', and may be characterized thus:

$$\begin{aligned} \text{II'. } & p > q, p - q \equiv 0 \pmod{3}, \alpha' \geq 1, m = p + 2q - 3 (\alpha' - 1), \pi = p - q \\ & - 3 (\alpha' - 1), \rho = 2q - 3 (\alpha' - 1), m' = \frac{1}{3} (p + 2q) - \alpha' + 1, \\ & f = R = R_i = R_c = 0. \end{aligned}$$

Class II' as here described, evidently includes the curves of Class I', as those curves were designated on page 279; hence, the latter class will no more be mentioned as a distinct group, but will be regarded as coming in with Class II'

This completes for the present the discussion of the curves $(p, q)_\alpha$, where $p - q \equiv 0 \pmod{3}$.

If $p - q \equiv 1 \pmod{3}$, it is known that the factor x^α , where α has the value $\frac{2}{3}(p - q - 1)$, can be rejected in every case from the equation obtained from $\phi = 0$ by the required substitutions. The conditions under which a factor $x^{\alpha + \alpha'}$, where $\alpha' \geq 1$, can be divided out from that equation are readily found by the method employed in the preceding case. For that purpose the exponent of x in each term of the equation in question must be as great as $\alpha + \alpha' = \frac{2}{3}(p - q - 1) + \alpha'$; i. e., it is necessary that $\frac{1}{3}\beta + \frac{2}{3}(p - q + \gamma)$, as found on page 282, be as great as $\frac{2}{3}(p - q - 1) + \alpha'$, which requires that

$$\beta \geq 3\alpha' - 2\gamma - 2, \text{ if } p - q + \gamma - \beta \equiv 0 \pmod{3}.$$

Likewise must

$$\beta \geq 3\alpha' - 2\gamma, \quad \text{if } p - q + \gamma - \beta \equiv 1 \pmod{3},$$

and

$$\beta \geq 3\alpha' - 2\gamma + 2, \text{ if } p - q + \gamma - \beta \equiv 2 \pmod{3}.$$

These conditions are entirely similar in form and application to those already discussed for the case where $p - q = 0 \pmod{3}$. Hence, since the terms in the group ϕ_{p-q} cannot all be wanting, the largest value is allowed to α' when the only non-vanishing coefficient in that group is $a_{p-q,0}$; this leaves a term $a_{p-q,0} \lambda^{p-q} \nu^q$ which becomes $a_{p-q,0} x^{p-q} s^q$; hence $\alpha + \alpha' \leq p - q$, which gives for α' an upper limit, $\alpha' \leq \frac{1}{3}(p - q + 2)$. There must also occur in the equation of $(p, q)_\alpha$ a term free from λ ; and the one giving the highest limit for $\beta + 2\gamma$, and, hence, for α' , is $a_{0,q} \mu^p$; this term becomes $a_{0,q} y^p$ and

$$a_{0,q} y^p = a_{0,q} x^{\frac{2}{3}p} z^{\frac{1}{3}p}, \quad \text{if } p \equiv 0 \pmod{3},$$

$$\text{or} \quad = a_{0,q} x^{\frac{2}{3}(p-1)} y z^{\frac{1}{3}(p-1)}, \quad \text{if } p \equiv 1 \pmod{3},$$

$$\text{or} \quad = a_{0,q} x^{\frac{2}{3}(p-2)} y^2 z^{\frac{1}{3}(p-2)}, \quad \text{if } p \equiv 2 \pmod{3},$$

as given on page 283. Consequently, $\alpha + \alpha'$ has an upper limit $\frac{2}{3}p$, $\frac{2}{3}(p - 1)$ or $\frac{2}{3}(p - 2)$ according as $p \equiv 0, 1$ or $2 \pmod{3}$; since $\alpha = \frac{2}{3}(p - q - 1)$, this gives as upper limits for α' that

$$\alpha' \leq \frac{2}{3}q + \frac{2}{3}, \text{ if } p \equiv 0 \pmod{3},$$

$$\text{or} \quad \leq \frac{2}{3}q, \quad \text{if } p \equiv 1 \pmod{3},$$

$$\text{or} \quad \leq \frac{2}{3}q - \frac{2}{3}, \text{ if } p \equiv 2 \pmod{3}.$$

Not only must the appropriate one of these last three conditions be obeyed by α' , but also in every case must $\alpha' \leq \frac{1}{3}(p - q + 2)$.

To obtain the equation of the curve $(p, q)_{\alpha'}$ from that of (p, q) , all terms of the latter, which have β falling below the limits found above, must vanish. If $\beta = 3\alpha' - 2\gamma + 1$, $p - q + \gamma - \beta = p - q - 3\alpha' + 3\gamma - 1 \equiv p - q - 1 \equiv 0 \pmod{3}$, and the coefficient $a_{3\alpha' - 2\gamma + 1, \gamma}$ is not required to vanish. If $\beta = 3\alpha' - 2\gamma - 1$, $p - q + \gamma - \beta = p - q - 3\alpha' + 3\gamma + 1 \equiv p - q + 1 \equiv 2 \pmod{3}$, and the coefficient $a_{3\alpha' - 2\gamma - 1, \gamma}$ must be made to vanish, under the requirements of the third condition for β above, viz., that $\beta \geq 3\alpha' - 2\gamma + 2$ if $p - q + \gamma - \beta \equiv 2 \pmod{3}$. These conditions require not only that $a_{\beta, \gamma} = 0$, when $\beta = 3\alpha' - 2\gamma - 1$, but also whenever $0 \leq \beta \leq 3\alpha' - 2\gamma - 3$. But if in any case these conditions require an $a_{\beta, \gamma}$ to vanish for a combination of values of β and γ which do not occur together in any coefficient of the equation $\phi = 0$, such a condition must not be counted in making up the number of restrictions to be actually imposed on the coefficients of the equation in question. The highest value of β reached in the conditions is $\beta = 3\alpha' - 2\gamma - 1$; that will surpass the highest power of λ appearing in the equation $\phi = 0$ when and only when $3\alpha' - 2\gamma - 1 > p - q + \gamma$; if α' takes its largest value $\frac{1}{3}(p - q + 2)$, the condition becomes $1 > 3\gamma$, which is unsatisfied except when $\gamma = 0$; hence, none of the conditions fails to be applicable when $\alpha' < \frac{1}{3}(p - q + 2)$, and only a single one of them is inapplicable when $\alpha' = \frac{1}{3}(p - q + 2)$; the coefficient, the vanishing of which the inapplicable condition demands, is $a_{3\alpha' - 2\gamma - 1, \gamma} = a_{p - q + 1, 0}$, and it is evident that no such coefficient ever appears in the equation of the curve (p, q) .

The number of conditions imposed on the coefficients of the equation of the curve (p, q) to make it become the equation of the curve $(p, q)_{\alpha'}$, in the case where $p - q \equiv 1 \pmod{3}$, is then $\sum_{\gamma=0}^{\lfloor \frac{3\alpha' - 1}{3} \rfloor} (3\alpha' - 2\gamma - 1)$, if $\alpha' < \frac{1}{3}(p - q + 2)$, and that number diminished by unity if $\alpha' = \frac{1}{3}(p - q + 2)$; this sum amounts to $\frac{1}{4}(3\alpha')^2$, if α' is even, and to $\frac{1}{4}(3\alpha' - 1)(3\alpha' + 1)$, if α' is odd; hence the number of conditions imposed, when $\alpha' < \frac{1}{3}(p - q + 2)$, is $\frac{1}{4}(3\alpha')^2$ or $\frac{1}{4}(9\alpha'^2 - 1)$, according as α' is even or odd; while, if $\alpha' = \frac{1}{3}(p - q + 2)$, the corresponding number is $\frac{1}{4}(3\alpha')^2 - 1$ or $\frac{1}{4}(9\alpha'^2 - 1) - 1$, according as α' is even or odd; but the number is always two, if $\alpha' = 1$.

The general equation $\phi = 0$, where $p - q \equiv 1 \pmod{3}$, has been found to represent a curve (p, q) of order $p + 2q$ which is cut from Σ by a surface S of order $\frac{1}{3}(p + 2q + 2)$, where the residual intersection consists of the cuspidal edge of Σ occurring twice, so that $f = 1$, $R = 2$, $R_i = 2$, and $R_c = 0$. If now the condition that $\alpha' \geq 1$ be imposed, the surface S' cutting the curve $(p, q)_{\alpha'}$ from Σ is clearly of order $\frac{1}{3}(p + 2q + 2) - \alpha'$. The conditions imposed to make the equation of S reducible by $x^{\alpha'}$ will, if $\alpha' \geq 1$, require that $(p, q)_{\alpha'}$ be a curve of lower order than the corresponding (p, q) , since the total intersection of S and Σ suffers a reduction by the cuspidal edge of Σ , occurring $3\alpha' \geq 3$ times, while the residual intersection for the curve (p, q) consists of that edge occurring only twice. But it is evident that the curve (p, q) is not in general subject to any other reduction, and the curve $(p, q)_{\alpha'}$ will consequently be in general a proper curve, and any residual intersection of S' and Σ will be made up entirely of the cuspidal edge of Σ . The coefficient $a_{\beta, \gamma}$, where $\beta = 3\alpha' - 2\gamma - 2$ and $p - q + \gamma - \beta \equiv 0 \pmod{3}$, does not vanish under the conditions of restriction here imposed; this condition is possible of satisfaction for β in every case, since $3\alpha' - 2\gamma - 2 \geq 0$, if $\gamma = 0$ and $\alpha' \geq 1$; hence there occurs in the equation of the surface S' the term derived from that one in $\phi = 0$ of the form

$$a_{3\alpha' - 2\gamma - 2, \gamma} \lambda^{3\alpha' - 2\gamma - 2} \mu^{p - q + \gamma - \beta} \nu^{q - \gamma};$$

this term, on performing the required substitutions and reductions, becomes

$$a_{3\alpha' - 2\gamma - 2, \gamma} x^{3\alpha' - 2\gamma - 2 + \frac{2}{3}(p - q) + 2\gamma - 2\alpha' + \frac{2}{3}z^{\frac{1}{3}(p - q) + \gamma - \alpha + \frac{2}{3}} s^{q - \gamma},$$

which is again

$$a_{3\alpha' - 2\gamma - 2, \gamma} x^{\frac{2}{3}(p - q - 1) + \alpha'} z^{\frac{1}{3}(p - q + 2) + \gamma - \alpha'} s^{q - \gamma}.$$

On rejecting the factor $x^{\alpha + \alpha'} = x^{\frac{2}{3}(p - q - 1) + \alpha'}$, this term becomes

$$a_{3\alpha' - 2\gamma - 2, \gamma} z^{\frac{1}{3}(p - q + 2) + \gamma - \alpha'} s^{q - \gamma},$$

which is of the zeroth degree in x, y ; consequently the surface S' does not contain the cuspidal edge of Σ at all, and the curve $(p, q)_{\alpha'}$ is accordingly a curve of total intersection of Σ and S' . In the case of the curve (p, q) , it has been found that $\pi = p - q$ and $\rho = 2q$; since the cuspidal edge of Σ has been rejected $3\alpha'$ times, and since the former residual intersection consisted of that

edge occurring twice, it follows that in the case of the curve $(p, q)_{\alpha'}$, where $p - q \equiv 1 \pmod{3}$, $\pi = p - q - 3\alpha' + 2$ and $\rho = 2q - 3\alpha' + 2$. The upper limits here involved for α' agree with those already found for the case in question. Here, again, the curves $(p, q)_{\alpha'}$ of total intersection have $\pi \equiv 0 \pmod{3}$. If both π and ρ have the value zero, then must the curve represented pass through neither the vertex of Σ nor the cusp of the infinite cubic, and hence the curve will be of more general character than the (p, q) of the same order, which has been found to pass a certain number of times through one or both of those points. Such a curve $(p, q)_{\alpha'}$ occurs when $\pi = p - q - 3\alpha' + 2 = \rho = 2q - 3\alpha' + 2 = 0$, which demands that $p = 3q$ and $\alpha' = \frac{2}{3}q + \frac{2}{3} = \frac{2}{3}p + \frac{2}{3}$, while $m = p$, $m' = \frac{1}{3}p$ and $2q = p - q \equiv 1 \pmod{3}$, making $q \equiv 2 \pmod{3}$; the $(6, 2)_2$ answering these requirements,—that $\alpha' = 2$, etc.,—is cut from Σ by the surface S' whose equation is determined as follows:

The general equation of the curve $(6, 2)$ is of the form:

$$\begin{aligned} a_{6,2}\lambda^6 + a_{5,2}\lambda^5\mu + a_{4,2}\lambda^4\mu^2 + a_{3,2}\lambda^3\mu^3 + a_{2,2}\lambda^2\mu^4 + a_{1,2}\lambda\mu^5 + a_{0,2}\mu^6 + (a_{5,1}\lambda^5 \\ + a_{4,1}\lambda^4\mu + a_{3,1}\lambda^3\mu^2 + a_{2,1}\lambda^2\mu^3 + a_{1,1}\lambda\mu^4 + a_{0,1}\mu^5)\nu + (a_{4,0}\lambda^4 + a_{3,0}\lambda^3\mu \\ + a_{2,0}\lambda^2\mu^2 + a_{1,0}\lambda\mu^3 + a_{0,0}\mu^4)\nu^2 = 0. \end{aligned}$$

The conditions to be imposed on the coefficients, in order that α' may have the value two, are found by the method used above to demand that $a_{1,2} = a_3 = a_{1,1} = a_{0,1} = a_{3,0} = a_{2,0} = a_{1,0} = a_{0,0} = 0$; this leaves the equation of the curve $(6, 2)_2$ which is of the form:

$$\begin{aligned} a_{6,2}\lambda^6 + a_{5,2}\lambda^5\mu + a_{4,2}\lambda^4\mu^2 + a_{3,2}\lambda^3\mu^3 + a_{2,2}\lambda^2\mu^4 + a_{0,2}\mu^6 \\ + a_{5,1}\lambda^5\nu + a_{4,1}\lambda^4\mu\nu + a_{2,1}\lambda^2\mu^3\nu + a_{4,0}\lambda^4\nu^2 = 0. \end{aligned}$$

Performing the required substitution, this becomes

$$\begin{aligned} a_{6,2}x^6 + a_{5,2}x^5y + a_{4,2}x^4y^2 + a_{3,2}x^3y^3 + a_{2,2}x^2y^4 + a_{0,2}y^6 \\ + a_{5,1}x^5s + a_{4,1}x^4ys + a_{2,1}x^2y^3s + a_{4,0}x^4s^2 = 0; \end{aligned}$$

changing y^3 to x^2z as many times as possible in each term of this equation, and rejecting the factor $x^{\alpha+\alpha'} = x^4$, there results the equation of the surface S' of the form:

$$\begin{aligned} a_{6,2}x^2 + a_{5,2}xy + a_{4,2}y^2 + a_{3,2}xz + a_{2,2}yz + a_{0,2}z^2 \\ + a_{5,1}xs + a_{4,1}ys + a_{2,1}zs + a_{4,0}s^2 = 0. \end{aligned}$$

This equation is that of the general quadric surface, and thus it is evident that the curve $(6, 2)_2$ is the most general curve which can be cut from Σ by a quadric surface.

Similarly, a general curve of order 15 is the $(15, 5)_4$ cut from Σ by a general surface of order 5; likewise, the $(24, 8)_6$, a curve of order 24, is the total intersection of the cone Σ with a general surface of order 8, the $(33, 11)_8$ of order 33 is the total intersection of Σ with a general surface of order 11, the $(42, 14)_{10}$, of order 42, is the total intersection of Σ with a general surface of order 14, etc., etc.

These special curves of total intersection $(p, q)_{\alpha'}$ will comprise Class III' and may be characterized thus:

$$\text{III'. } p - q \equiv 1 \pmod{3}, \alpha' \geq 1, m = p + 2q - 3\alpha' + 2, \pi = p - q - 3\alpha' + 2, \\ \rho = 2q - 3\alpha' + 2, m' = \frac{1}{3}(p + 2q + 2) - \alpha', f = R = R_i = R_c = 0.$$

If $p - q \equiv 2 \pmod{3}$, it has been proved that x^α , where $\alpha = \frac{2}{3}(p - q - 2)$, is a factor of all the terms of the equation obtained from $\phi = 0$ by the required substitutions. The conditions under which $x^{\alpha+\alpha'}$, where $\alpha' \geq 1$, becomes a factor in the same equation, are readily found by the method used in the two cases already discussed. For the occurrence of such a factor, it is necessary that the power of x in each term be as great as $\alpha + \alpha' = \frac{2}{3}(p - q - 2) + \alpha'$; i. e., it is necessary that $\frac{1}{3}\beta + \frac{2}{3}(p - q + \gamma)$, as given on page 282, be as great as $\frac{2}{3}(p - q - 2) + \alpha'$, which requires that

$$\beta \geq 3\alpha' - 2\gamma - 4, \text{ if } p - q + \gamma - \beta \equiv 0 \pmod{3}.$$

Likewise must

$$\beta \geq 3\alpha' - 2\gamma - 2, \text{ if } p - q + \gamma - \beta \equiv 1 \pmod{3},$$

or

$$\beta \geq 3\alpha' - 2\gamma, \quad \text{if } p - q + \gamma - \beta \equiv 2 \pmod{3}.$$

As in the preceding cases, the terms in the group ϕ_{p-q} cannot all be wanting, and the highest value of α' is allowed when the only coefficient remaining in that group is $a_{p-q,0}$; this leaves the term $a_{p-q,0}\lambda^{p-q}\nu^q$ present, which term becomes $a_{p-q,0}x^{p-q}s^q$; hence $\alpha + \alpha' \leq p - q$, whence $\alpha' \leq p - q - \alpha$, giving as an upper limit for α' that $\alpha' \leq \frac{1}{3}(p - q + 4)$. The restricted equation $\phi = 0$ must contain a term free from λ and hence of the form $a_{0,\gamma}\mu^{p-q+\gamma}\nu^{q-\gamma}$; this term will permit

the highest value of $\beta + 2\gamma$, and thus of α' , when $\gamma = q$; then it is of the form $\alpha_{0,q}\mu^p$, which becomes $\alpha_{0,q}y^p$; this reduces to

$$\alpha_{0,q} x^{\frac{1}{3}p} z^{\frac{1}{3}p}, \quad \text{if } p \equiv 0 \pmod{3},$$

$$\text{or} \quad \alpha_{0,q} x^{\frac{1}{3}(p-1)} y z^{\frac{1}{3}(p-1)}, \quad \text{if } p \equiv 1 \pmod{3},$$

$$\text{or} \quad \alpha_{0,q} x^{\frac{1}{3}(p-2)} y^2 z^{\frac{1}{3}(p-2)}, \quad \text{if } p \equiv 2 \pmod{3},$$

as seen on page 283. Consequently, $\alpha + \alpha'$ has as an upper limit $\frac{2}{3}p$, $\frac{2}{3}(p-1)$ or $\frac{2}{3}(p-2)$, according as $p \equiv 0, 1$ or $2 \pmod{3}$; since $\alpha = \frac{2}{3}(p-q-2)$, this gives as upper limits for α' the requirement that $\alpha' \leq \frac{2}{3}q + \frac{1}{3}$, $\frac{2}{3}q + \frac{2}{3}$ or $\frac{2}{3}q$ according as $p \equiv 0, 1$ or $2 \pmod{3}$. Not only must the appropriate one of these last three conditions be satisfied by α' , but also, in every case, the condition found above,—that $\alpha' \leq \frac{1}{3}(p-q+4)$,—must be fulfilled.

The equation of (p, q) is made to become the equation of $(p, q)_{\alpha'}$ by rejecting from it all those terms in which β falls below the limits found above for it. If $\beta = 3\alpha' - 2\gamma - 1$, then $p - q + \gamma - \beta = p - q + \gamma - (3\alpha' - 2\gamma - 1) = p - q + 3(\gamma - \alpha') + 1 \equiv 1 \pmod{3}$, and no coefficient $a_{\beta,\gamma}$ need vanish for such a value of β . If $\beta = 3\alpha' - 2\gamma - 3$, then $p - q + \gamma - \beta = p - q + \gamma - (3\alpha' - 2\gamma - 3) = p - q + 3(\gamma - \alpha' + 1) \equiv 2 \pmod{3}$, and the coefficient $a_{3\alpha' - 2\gamma - 3, \gamma}$ must vanish. Likewise must every coefficient $a_{\beta,\gamma}$ vanish where $0 \leq \beta \leq 3\alpha' - 2\gamma - 5$. It may happen here that β, γ and α' may satisfy the conditions for such an arrangement of values of β and γ as does not occur in the equation $\phi = 0$, in which case the condition in question should not be counted in making up the total number of conditions to be imposed on that equation. Such an inapplicable condition occurs when $\beta = 3\alpha' - 2\gamma - 3$ is greater than the power of λ occurring with the value of γ in question, i. e., when $3\alpha' - 2\gamma - 3 > p - q + \gamma$; since $\alpha' \leq \frac{1}{3}(p-q+4)$, such a case can arise only when $p - q + 4 - 2\gamma - 3 > p - q + \gamma$, i. e., when $1 > 3\gamma$, and this is satisfied only when $\gamma = 0$; hence, one of the above conditions is inapplicable when and only when $\alpha' = \frac{1}{3}(p-q+4)$ and $\gamma = 0$. The coefficient required to vanish in this case is $a_{p-q+1,0}$, and no such coefficient appears in the equation $\phi = 0$.

The number of conditions imposed on the coefficients of the equation $\phi = 0$ to obtain the equation of the curve $(p, q)_{\alpha'}$ in the case in question, where $p - q$

$\equiv 2 \pmod{3}$, is $\sum_{\gamma=0}^{[\frac{1}{3}(\alpha'-1)]} (3\alpha' - 2\gamma - 3)$, if $\alpha' < \frac{1}{3}(p-q+4)$; this sum amounts to

$\frac{1}{4}(3\alpha' - 2)^2$, if α' is even, and to $\frac{3}{4}(\alpha' - 1)(3\alpha' - 1)$ if α' be odd and greater than unity, but to unity, if $\alpha' = 1$; hence, the number of conditions imposed, when $\alpha' = \frac{1}{3}(p - q + 4)$, is $\frac{1}{4}(3\alpha' - 2)^2 - 1$, if α' is even, and $\frac{3}{4}(\alpha' - 1)(3\alpha' - 1) - 1$, if α' is odd; α' cannot take the value unity in this case, since $\alpha' = \frac{1}{3}(p - q + 4) = 1$ requires that $p - q = -1$, which is impossible, p always being at least as great as q in the geometry on Σ .

The general equation $\phi = 0$, for any possible values of p and q , where $p - q \equiv 2 \pmod{3}$, has been found to represent a curve (p, q) of order $p + 2q$, cut from Σ by a surface S of order $\frac{1}{3}(p + 2q + 4)$, the residual intersection consisting of the cuspidal edge of Σ occurring 4 times, while $f = 2$, $R = 4$, $R_i = 4$, $R_c = 0$. If $\alpha' = 1$, the equation of the resultant surface of intersection S' must be of degree as great as unity in x, y , since the equation of the surface S was found to be of degree two in x, y , and the factor $x^{\alpha'} = x$ only has been rejected to obtain the former equation from the latter; the coefficient $a_{0,0} = 0$, in accordance with the conditions imposed here, and that is the only coefficient made to vanish in this case; some term in the variables μ, ν of the form $a_{0,\gamma} \mu^{p-q+\gamma} \nu^{q-\gamma}$, where $\gamma \geq 1$, must appear, otherwise the equation of the curve $(p, q)_1$ would be reducible at once by λ ; the term $a_{0,\gamma} \mu^{p-q+\gamma} \nu^{q-\gamma}$, under the required substitutions, takes the form

$$a_{0,\gamma} x^{\frac{1}{3}(p-q+\gamma)} z^{\frac{1}{3}(p-q+\gamma)} s^{q-\gamma}, \quad \text{if } p - q + \gamma \equiv 0 \pmod{3},$$

$$a_{0,\gamma} x^{\frac{1}{3}(p-q+\gamma-1)} y z^{\frac{1}{3}(p-q+\gamma-1)} s^{q-\gamma}, \quad \text{if } p - q + \gamma \equiv 1 \pmod{3},$$

or
$$a_{0,\gamma} x^{\frac{1}{3}(p-q+\gamma-2)} y^2 z^{\frac{1}{3}(p-q+\gamma-2)} s^{q-\gamma}, \quad \text{if } p - q + \gamma \equiv 2 \pmod{3}.$$

The condition that $p - q \equiv 2 \pmod{3}$ demands that γ take a value at least as great as two in some term of the equation; $\gamma = 2$ makes $p - q + \gamma \equiv 1 \pmod{3}$, giving a term $a_{0,2} x^{\frac{1}{3}(p-q+1)} y z^{\frac{1}{3}(p-q+1)} s^{q-2}$; the rejection of the factor $x^{\alpha+\alpha'} = x^{\frac{1}{3}(p-q+1)}$ leaves this term of degree unity in x, y and containing the variable y ; hence the surface S' contains the cuspidal edge of Σ twice and only twice; thus the residual intersection of Σ and S' is made up in this case of the cuspidal edge of Σ occurring twice; consequently, $m' = \frac{1}{3}(p + 2q + 4) - 1 = \frac{1}{3}(p + 2q + 1)$ and $m = p + 2q - 1$, while $f = 1$, $R = 2$, $R_i = 2$, $R_c = 0$. The curve (p, q) was found in the cases in question to have $\pi = p - q$; that curve has now been broken up into the curve $(p, q)_1$ and the cuspidal edge occurring once, since the total order in each case is $p + 2q$; hence the curve $(p, q)_1$ has $\pi = p - q - 1$,

and, similarly, $\rho = 2q - 1$. Curves $(p, q)_1$, which are none of them curves of total intersection, when $p - q \equiv 2 \pmod{3}$, will be grouped as Class IV'', and may be characterized thus:

$$\text{IV''}. \quad p - q \equiv 2 \pmod{3}, \alpha' = 1, m = p + 2q - 1, \pi = p - q - 1, \\ \rho = 2q - 1, m' = \frac{1}{3}(p + 2q + 1), f = 1, R = 2, R_i = 2, R_c = 0,$$

If $\alpha' \geq 2$, the surface S' must be of order less by α' than the order of S , i. e. $m' = \frac{1}{3}(p + 2q + 4) - \alpha'$. As in the preceding cases, the curve $(p, q)_{\alpha'}$ will be in general, a proper curve, and the entire residual intersection will be made up of the cuspidal edge of Σ occurring the necessary number of times. The coefficient $a_{\beta, \gamma}$, where $\beta = 3\alpha' - 2\gamma - 4$ and $p - q + \gamma - \beta = p - q + 3\gamma - 3\alpha' + 4 \equiv 0 \pmod{3}$ does not vanish under the conditions imposed; this condition is possible of satisfaction in every case in question, since $3\alpha' - 2\gamma - 4 \geq 0$ so long as $\alpha' \geq 2$; hence there occurs in the equation of the curve $(p, q)_{\alpha'}$ a term or terms of the form $a_{\beta, \gamma} \lambda^{\beta} \mu^{p-q+\gamma-\beta} \nu^{q-\gamma} = a_{3\alpha'-2\gamma-4, \gamma} \lambda^{3\alpha'-2\gamma-4} \mu^{p-q+3\gamma-3\alpha'+4} \nu^{q-\gamma}$; this term becomes $a_{3\alpha'-2\gamma-4, \gamma} x^{3\alpha'-2\gamma-4+\frac{2}{3}(p-q)+2\gamma-2\alpha'+\frac{2}{3}(p-q+4)+\gamma-\alpha'} s^{q-\gamma}$, and on rejecting $x^{\alpha'+\alpha'} = x^{\frac{2}{3}(p-q-2)+\alpha'}$, this becomes $a_{3\alpha'-2\gamma-4, \gamma} z^{\frac{1}{3}(p-q+4)+\gamma-\alpha'} s^{q-\gamma}$, which involves neither x nor y ; hence S' does not contain the cuspidal edge of Σ , there is no residual intersection, and consequently all curves $(p, q)_{\alpha'}$ where $\alpha' \geq 2$ and $p - q \equiv 2 \pmod{3}$, are curves of total intersection of Σ and a surface S' . The curve $(p, q)_{\alpha'}$ is of order $m = 3[\frac{1}{3}(p + 2q + 4) - \alpha'] = p + 2q - 3\alpha' + 4$, which is less by $3\alpha' - 4$ than the order of the curve (p, q) ; this reduction in order has resulted from rejecting the cuspidal edge of Σ $3\alpha' - 4$ times; hence the number of times the curve $(p, q)_{\alpha'}$ passes through the vertex of Σ must be less than in the case of (p, q) by that same number; thus it is found here that $\pi = p - q - 3\alpha' + 4$. In like manner it is seen that $\rho = 2q - 3\alpha' + 4$. Here again $\pi \equiv 0 \pmod{3}$. If both π and ρ have the value zero, then must the curve $(p, q)_{\alpha'}$ contain neither the vertex of Σ nor the cusp of the infinite cubic, and hence is a more general curve than a (p, q) of the same order, which must always contain one or both of the two points in question. Such a curve $(p, q)_{\alpha'}$ is given when $\pi = p - q - 3\alpha' + 4 = \rho = 2q - 3\alpha' + 4 = 0$, which requires that $p = 3q$ and $\alpha' = \frac{2}{3}q + \frac{4}{3} = \frac{2}{3}p + \frac{4}{3}$, while $m = p$, $m' = \frac{1}{3}p$, and $2q = p - q \equiv 2 \pmod{3}$, making $q \equiv 1 \pmod{3}$; the $(3, 1)_1$ answering these conditions,

—that $\alpha' = 2$, etc.,—is cut from Σ by the surface S' whose equation is found as follows: The most general equation of a curve (3, 1) is of the form

$$a_{3,1}\lambda^3 + a_{2,1}\lambda^2\mu + a_{1,1}\lambda\mu^2 + a_{0,1}\mu^3 + (a_{2,0}\lambda^2 + a_{1,0}\lambda\mu + a_{0,0}\mu^2)\nu = 0.$$

The conditions to be imposed on the coefficients of this equation, in order that α' may take its greatest value two, are found by the method used above to be in number $\frac{1}{4}(3\alpha' - 2)^2 - 1 = \frac{1}{4}(6 - 2)^2 - 1 = 3$; these conditions are satisfied by making $a_{1,1} = a_{1,0} = a_{0,0} = 0$, the other coefficients remaining; this leaves the equation of the curve (3, 1)₁, which is then of the form

$$a_{3,1}\lambda^3 + a_{2,1}\lambda^2\mu + a_{0,1}\mu^3 + a_{2,0}\lambda^2\nu = 0.$$

Performing the required substitution, this becomes

$$a_{3,1}x^3 + a_{2,1}x^2y + a_{0,1}y^3 + a_{2,0}x^2s = 0,$$

and the change of y^3 to x^2z allows the rejection of the factor x^2 , so that the equation of the surface S' is found to be

$$a_{3,1}x + a_{2,1}y + a_{0,1}z + a_{2,0}s = 0,$$

the most general equation of the plane; hence the curve (3,1)₁ is the most general curve which can be cut from Σ by a plane.

Similarly, the (12,4)₄ of order 12 is the total intersection of Σ with the general quartic surface; the (21,7)₆, a curve of order 21, is the total intersection of Σ with the general surface of order 7; the (30,10)₈, a curve of order 30, is the total intersection of Σ with the general surface of order 10; etc., etc.

These special curves $(p, q)_{\alpha'}$ of total intersection will form Class IV' and may be characterized thus:

$$\text{IV'. } p - q \equiv 2 \pmod{3}, \alpha' \geq 2, m = p + 2q - 3\alpha' + 4, \pi = p - q - 3\alpha' + 4, \\ \rho = 2q - 3\alpha' + 4, m' = \frac{1}{3}(p + 2q + 4) - \alpha', f = R = R_i = R_c = 0.$$

Letting C denote the number of conditions imposed on the equation of general form to obtain the restricted equation $\phi = 0$, the results derived in the preceding pages may be collected and arranged thus:

Curves (p, q) Given by Equations of General Form.

1. Curves of Total Intersection.

I. $p = q, m = 3p, \pi = 0, \rho = 2q = 2p, m' = p, f = R = R_i = R_e = 0.$

2. Curves of Partial Intersection.

II. $p > q, p - q \equiv 0 \pmod{3}, m = p + 2q, \pi = p - q, \rho = 2q,$
 $m' = \frac{1}{3}(p + 2q) + 1, f = 1, R = 3, R_i = 2, R_e = 1.$

III. $p - q \equiv 1 \pmod{3}, m = p + 2q, \pi = p - q, \rho = 2q,$
 $m' = \frac{1}{3}(p + 2q + 2), f = 1, R = 2, R_i = 2, R_e = 0.$

IV. $p - q \equiv 2 \pmod{3}, m = p + 2q, \pi = p - q, \rho = 2q,$
 $m' = \frac{1}{3}(p + 2q + 4), f = 2, R = 4, R_i = 4, R_e = 0.$

Curves (p, q) Given by Equations of Special Form.

1. Curves of Total Intersection.

II'. $p > q, p - q \equiv 0 \pmod{3}, \alpha' \geq 1, m = p + 2q - 3(\alpha' - 1),$
 $\pi = p - q - 3(\alpha' - 1), \rho = 2q - 3(\alpha' - 1), m' = \frac{1}{3}(p + 2q) - \alpha' + 1,$
 $f = R = R_i = R_e = 0; \text{ here } \alpha' \leq \frac{1}{3}(p - q) + 1,$

and also

$$\alpha' \leq \begin{cases} \frac{2}{3}q + 1, & \text{if } p \equiv 0 \pmod{3}, \\ \frac{2}{3}q + \frac{1}{3}, & \text{if } p \equiv 1 \pmod{3}, \\ \frac{2}{3}q - \frac{1}{3}, & \text{if } p \equiv 2 \pmod{3}, \end{cases}$$

and

$$C = \begin{cases} \frac{3}{4}\alpha'(3\alpha' - 2) + 1, & \text{if } \alpha' < \frac{1}{3}(p - q) + 1 \text{ and even,} \\ \frac{1}{4}(3\alpha' - 1)^2, & \text{if } \alpha' < \frac{1}{3}(p - q) + 1 \text{ and odd,} \\ \text{or} & \text{if } \alpha' = \frac{1}{3}(p - q) + 1 = 1, \\ \frac{3}{4}\alpha'(3\alpha' - 2), & \text{if } \alpha' = \frac{1}{3}(p - q) + 1 \text{ and even,} \\ \frac{1}{4}(3\alpha' - 1)^2 - 1, & \text{if } \alpha' = \frac{1}{3}(p - q) + 1 > 1 \text{ and odd.} \end{cases}$$

III'. $p - q \equiv 1 \pmod{3}, \alpha' \geq 1, m = p + 2q - 3\alpha' + 2, \pi = p - q - 3\alpha' + 2,$
 $\rho = 2q - 3\alpha' + 2, m' = \frac{1}{3}(p + 2q + 2) - \alpha', f = R = R_i = R_e = 0; \text{ here}$
 $\alpha' \leq \frac{1}{3}(p - q + 2),$

and also

$$\alpha' \leq \begin{cases} \frac{2}{3}q + \frac{2}{3}, & \text{if } p \equiv 0 \pmod{3}, \\ \frac{2}{3}q, & \text{if } p \equiv 1 \pmod{3}, \\ \frac{2}{3}q - \frac{2}{3}, & \text{if } p \equiv 2 \pmod{3}, \end{cases}$$

and

$$C = \begin{cases} (\frac{3}{2}\alpha')^2, & \text{if } \alpha' < \frac{1}{3}(p-q+2) \text{ and even,} \\ \frac{1}{4}(9\alpha'^2-1), & \text{if } \alpha' < \frac{1}{3}(p-q+2) \text{ and odd,} \\ \text{or} & \text{if } \alpha' = \frac{1}{3}(p-q+2) = 1, \\ (\frac{3}{2}\alpha')^2 - 1, & \text{if } \alpha' = \frac{1}{3}(p-q+2) \text{ and even,} \\ \frac{1}{4}(9\alpha'^2-1) - 1, & \text{if } \alpha' = \frac{1}{3}(p-q+2) \text{ and odd.} \end{cases}$$

IV'. $p-q \equiv 2 \pmod{3}$, $\alpha' \geq 2$, $m = p + 2q - 3\alpha' + 4$, $\pi = p - q - 3\alpha' + 4$,
 $\rho = 2q - 3\alpha' + 4$, $m' = \frac{1}{3}(p + 2q + 4) - \alpha'$, $f = R = R_i = R_e = 0$; here
 $\alpha' \leq \frac{1}{3}(p - q + 4)$,

and also

$$\alpha' \leq \begin{cases} \frac{2}{3}q + \frac{4}{3}, & \text{if } p \equiv 0 \pmod{3}, \\ \frac{2}{3}q + \frac{2}{3}, & \text{if } p \equiv 1 \pmod{3}, \\ \frac{2}{3}q, & \text{if } p \equiv 2 \pmod{3}, \end{cases}$$

and

$$C = \begin{cases} \frac{1}{4}(3\alpha' - 2)^2, & \text{if } \alpha' < \frac{1}{3}(p - q + 4) \text{ and even,} \\ \frac{3}{4}(\alpha' - 1)(3\alpha' - 1) & \text{if } \alpha' < \frac{1}{3}(p - q + 4) \text{ and odd,} \\ \frac{1}{4}(3\alpha' - 2)^2 - 1, & \text{if } \alpha' = \frac{1}{3}(p - q + 4) \text{ and even,} \\ \frac{3}{4}(\alpha' - 1)(3\alpha' - 1), & \text{if } \alpha' = \frac{1}{3}(p - q + 4) \text{ and odd} \\ \text{(the case where } \alpha' = \frac{1}{3}(p - q + 4) = 1 \text{ coming under IV'').} \end{cases}$$

2. Curves of Partial Intersection.

IV''. $p - q \equiv 2 \pmod{3}$, $\alpha' = 1$, $m = p + 2q - 1$, $\pi = p - q - 1$,
 $\rho = 2q - 1$, $m' = \frac{1}{3}(p + 2q + 1)$, $f = 1$, $R = 2$, $R_i = 2$, $R_e = 0$

and $C = 1$.

The following table gives the results found by the formulæ given above for the cases of all proper curves (p, q) for values of p and q as far as the curve (8, 6).

TABLE.

p	q	$p-q$	α'	C	m	π	ρ	Class	m'	f	R	R_i	R_c	p	q	$p-q$	α'	C	m	π	ρ	Class	m'	f	R	R_i	R_c			
1	0	1	0	0	1	1	0	III.	1	1	2	2	0	6	4	2	0	0	14	2	8	IV.	6	2	4	4	0			
1	1	0	0	0	3	0	2	I.	1	0	0	0	0				1	1	13	1	7	IV."	5	1	2	2	0			
2	1	1	0	0	4	1	2	III.	2	1	2	2	0				2	3	12	0	6	IV.'	4	0	0	0	0			
2	2	0	0	0	6	0	4	I.	2	0	0	0	0	6	5	1	0	0	16	1	10	III.	6	1	2	2	0			
3	1	2	0	0	5	2	2	IV.	3	2	4	4	0				1	2	15	0	9	III.'	5	0	0	0	0			
			1	1	4	1	1	IV."	2	1	2	2	0	6	6	0	0	0	18	0	12	I.	6	0	0	0	0			
			2	3	3	0	0	IV.'	1	0	0	0	0	7	1	6	0	0	9	6	2	II.	4	1	3	2	1			
3	2	1	0	0	7	1	4	III.	3	1	2	2	0				1	1	9	6	2	II.'	3	0	0	0	0			
			1	2	6	0	3	III.'	2	0	0	0	0	7	2	5	0	0	11	5	4	IV.	5	2	4	4	0			
3	3	0	0	0	9	0	6	I.	3	0	0	0	0				1	1	10	4	3	IV."	4	1	2	2	0			
4	1	3	0	0	6	3	2	II.	3	1	3	2	1				2	4	9	3	2	IV.'	3	0	0	0	0			
			1	1	6	3	2	II.'	2	0	0	0	0	7	3	4	0	0	13	4	6	III.	5	1	2	2	0			
4	2	2	0	0	8	2	4	IV.	4	2	4	4	0				1	2	12	3	5	III.'	4	0	0	0	0			
			1	1	7	1	3	IV."	3	1	2	2	0				2	8	9	0	2	III.'	3	0	0	0	0			
			2	3	6	0	2	IV.'	2	0	0	0	0	7	4	3	0	0	15	3	8	II.	6	1	3	2	1			
4	3	1	0	0	10	1	6	III.	4	1	2	2	0				1	1	15	3	8	II.'	5	0	0	0	0			
			1	2	9	0	5	III.'	3	0	0	0	0				2	6	12	0	5	II.'	4	0	0	0	0			
4	4	0	0	0	12	0	6	I.	4	0	0	0	0	7	5	2	0	0	17	2	10	IV.	7	2	4	4	0			
5	1	4	0	0	7	4	2	III.	3	1	2	2	0				1	1	16	1	9	IV."	6	1	2	2	0			
5	2	3	0	0	9	3	4	II.	4	1	3	2	1				2	3	15	0	8	IV.'	5	0	0	0	0			
			1	1	9	3	4	II.'	3	0	0	0	0	7	6	1	0	0	19	1	12	III.	7	1	2	2	0			
5	3	2	0	0	11	2	6	IV.	5	2	4	4	0				1	2	18	0	11	III.'	6	0	0	0	0			
			1	1	10	1	5	IV."	4	1	2	2	0	7	7	0	0	0	21	0	14	I.	7	0	0	0	0			
			2	3	9	0	4	IV.'	3	0	0	0	0	8	1	7	0	0	10	7	2	III.	4	1	2	2	0			
5	4	1	0	0	13	1	8	III.	5	1	2	2	0				8	2	6	0	0	12	6	4	II.'	4	0	0	0	0
			1	2	12	0	7	III.'	4	0	0	0	0				1	1	12	6	4	II.'	4	0	0	0	0			
5	5	0	0	0	15	0	10	I.	5	0	0	0	0				8	3	5	0	0	14	5	6	IV.	6	2	4	4	0
6	1	5	0	0	8	5	2	IV.	4	2	4	4	0				1	1	13	4	5	IV."	5	1	2	2	0			
			1	1	7	4	1	IV."	3	1	2	2	0				2	4	12	3	4	IV.'	4	0	0	0	0			
			2	4	6	3	0	IV.'	2	0	0	0	0	8	4	4	0	0	16	4	8	III.	6	1	2	2	0			
6	2	4	0	0	10	4	4	III.	4	1	2	2	0				1	2	15	3	7	III.'	5	0	0	0	0			
			1	2	9	3	3	III.'	3	0	0	0	0				2	8	12	0	4	III.'	4	0	0	0	0			
			2	8	6	0	0	III.'	2	0	0	0	0	8	5	3	0	0	18	3	10	II.	7	1	3	2	1			
6	3	3	0	0	12	3	6	II.	5	1	3	2	1				1	1	18	3	10	II.'	6	0	0	0	0			
			1	1	12	3	6	II.'	4	0	0	0	0				2	6	15	0	7	II.'	5	0	0	0	0			
			2	6	9	0	3	II.'	4	0	0	0	0	8	6	2	0	0	20	2	12	IV.	8	2	4	4	0			